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# CONTENTS



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# ON THE REPRESENTATIONS OF THE SYMMETRIC GROUP.*

By F. D. Murnaghan.

**Introduction.** The representation theory of the symmetric group (group of $n!$ permutations of $n$ letters) was initiated by Frobenius some forty years ago and was developed in the, now classical, papers of Schur and Young. More recently Littlewood and Richardson (**13**) have discussed in detail the problem of the construction of the character table and have used a recurrence formula (passing from the symmetric group on $(n-1)$ letters to the symmetric group on $n$ letters) due to Schur in order to determine the characters of those classes of permutations which contain *at least one* unary cycle ($=$ fixed letter). We show in the present paper that this recurrence formula of Schur is but a special case of a general recurrence formula by means of which the characters of a class containing *at least one cycle* on $p$ letters $(1 \leq p < n)$ may be determined from the characters of the symmetric group on $n-p$ letters. As the characters of the class containing just one cycle (on $n$ letters) are trivially evident (as was pointed out by Frobenius) the construction of the character tables for the various symmetric groups $(n = 1, 2, 3, \cdots)$ is a routine matter demanding only paper and ink; and the easiest characters to calculate are those of classes containing cycles on the greatest number of letters.

The representation theory of the symmetric group is of importance in nuclear physics and in this connection the following two questions are of particular significance.

1. If we imagine the $n$ letters, whose permutations constitute the symmetric group, to be divided up in compartments or boxes containing, respectively, $\lambda_1, \lambda_2, \cdots, \lambda_k$ letters (so that $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$) we obtain a

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subgroup of the symmetric group by considering those permutations which do not send any letter out of its box. The cosets (right or left) of this subgroup furnish a representation (whose elements are permutation matrices) of the symmetric group; this representation is, in general, reducible and it is important to determine its analysis into irreducible components. The solution of this question is quite simple and is well known when $k=2$, i. e. when there are but two boxes. We give the solution in the general case.

2. The "direct product" of an irreducible representation of the symmetric group on $n$ letters by an irreducible representation of the symmetric group on $m$ letters furnishes a representation, in general reducible, of the symmetric group on $n+m$ letters and it is important to determine the analysis of this reducible representation into its irreducible components. We show how to do this, without having to use the character tables, and record the results for all values of $n$ and $m$ for which $n+m \leqq 9$.

In the hope of making the theory of the representations of the symmetric group more accessible to workers in nuclear physics we have made the following account somewhat self-contained. The original papers of Frobenius, and particularly those of Schur, arouse in a persevering reader an emotion akin to that inspired by one of the great symphonies; but they are by no means easy reading and we hope that a somewhat elementary orchestration may acquaint a larger audience with the work of the masters. It is a pleasure to here record our obligation, amongst others, to Professor Wedderburn for a pregnant remark which materially aided and simplified our treatment of the problem 1 of the preceding paragraph.

**1. The characteristics of a finite group.** Let $G$ be a finite group of order $N$; its elements will fall into $r$ classes (of conjugate elements) $C_1, \cdots, C_r$ such that if $g_r \subset C_r$ each element of $C_r$ is of the form $g g_r g^{-1}$, $g \subset G$. We denote by $N_p$ the number of elements in the $p$-th class $C_p$ so that, $C_1$ being the class consisting of the identity element $g_1$, $N_1 = 1$ and $N_1 + N_2 + \cdots + N_r = N$. By a representation of $G$ is meant a linear group homomorphic to $G$ and it is well known that $G$ possesses exactly $r$ non-equivalent irreducible representations $\Gamma_1, \cdots, \Gamma_r$. These are distinguished from one another by their characters and we denote by $\chi_p{}^q$ the character of $\Gamma_p$ associated with the class $C_q$; i. e. $\chi_p{}^q = \chi_p(g_q)$ where $g_q \subset C_q$. These characters satisfy certain fundamental orthogonality relations which are most conveniently stated as follows. If $a(g)$, $b(g)$ are any two complex valued functions defined over $G$ we denote by $(\boldsymbol{a} \cdot \boldsymbol{b})$ the average of the product $a(g)\bar{b}(g)$ over $G$ (the superposed bar denoting, as usual, the complex conjugate):

$$(\boldsymbol{a}\cdot\boldsymbol{b})=\frac{1}{N}\sum_{g}a(g)\bar{b}(g)$$

We shall be interested only in the case where $a(g)$ and $b(g)$ are class functions: $a(g_q)=a^q$; $b(g_q)=b^q$; $g_q\subset C_q$ and then $(\boldsymbol{a}\cdot\boldsymbol{b})=\frac{1}{N}\sum_{q}N_q a^q\bar{b}^q$. Then the orthogonality relations referred to are

$$(1)\qquad (\boldsymbol{\chi}_p\cdot\boldsymbol{\chi}_q)=0;\ p\neq q;\ (\boldsymbol{\chi}_p\cdot\boldsymbol{\chi}_p)=1;\qquad (p=1,2,\cdots,r).$$

These imply that any class function $\boldsymbol{a}$ is a linear combination of the $r$ functions $\boldsymbol{\chi}_p$: $\boldsymbol{a}=c^\alpha\boldsymbol{\chi}_\alpha$ (usual summation convention) where $c^p=(\boldsymbol{a}\cdot\boldsymbol{\chi}_p)$. Denoting by $\boldsymbol{s}$ a class function whose $r$ components $s^q$, $(q=1,\cdots,r)$, are indeterminates the expression $\phi(\boldsymbol{s})=(\boldsymbol{s}\cdot\boldsymbol{\chi}_p)=\frac{1}{N}\sum_{q}N_q\bar{\chi}_p{}^q s^q$ is called the *characteristic* of the irreducible representation $\Gamma_p$ and the characters $\chi_p{}^q$ are called the components of this characteristic. Any representation of $G$ is of the form $c^\alpha\Gamma_\alpha$ where the coefficients $c^p$ are integers (positive or zero) and the characters of this representation are $c^\alpha\chi_\alpha{}^q$; the corresponding expression $c^\alpha\phi_\alpha(\boldsymbol{s})$ being termed the characteristic of the given representation (with components $c^\alpha\chi_\alpha{}^q$). When all the coefficients $c^p$ vanish save one which is unity, so that the representation is irreducible, the characteristic is termed *simple*; otherwise it is called *compound* so that the characteristic of a reducible representation is compound. It is occasionally convenient to allow the coefficients $c^p$ in the expression $c^\alpha\phi_\alpha(\boldsymbol{s})$ to take negative integral as well as positive integral or zero values and then $c^\alpha\phi_\alpha(\boldsymbol{s})$ is termed a generalized characteristic, it being clearly understood that if any of the coefficients $c^p$ are negative the components $c^\alpha\chi_\alpha{}^q$ of the generalised characteristic are not the characters of any representation.

If $(\boldsymbol{s}\cdot c^\alpha\boldsymbol{\chi}_\alpha)$ is a generalised characteristic with components $c^\alpha\chi_\alpha{}^q=a^q$ we see at once from the orthogonality relations (1) that $(c^\alpha\chi_\alpha\cdot c^\alpha\chi_\alpha)=\sum_{\alpha=1}^{r}(c^\alpha)^2$ so that $(\boldsymbol{a}\cdot\boldsymbol{a})=1$ implies all the coefficients $c^p$ zero save one which $=\pm 1$. The generalised characteristic will, therefore, be simple if $(\boldsymbol{a}\cdot\boldsymbol{a})=1$ and if $a^1$, the coefficient of $s^1\frac{1}{N}$, $>0$ (for the coefficient of $s^1$ in a simple characteristic yields, on multiplication by $N$, the dimension of the corresponding irreducible representation of $G$, and is, accordingly, positive). Amongst the irreducible representations $(\Gamma_1,\cdots,\Gamma_r)$ of *any* finite group occurs the identity representation $\Gamma_1$ (in which to each element $g\subset G$ there corresponds the one

dimensional unit matrix) and the associated simple characteristic is called the *principal* characteristic of $G$; its explicit expression is $\phi_1(\boldsymbol{s}) = \frac{1}{N}\sum_{q=1}^{r} N_q s^q$ so that the coefficient of $s^q$ in $\phi_1(\boldsymbol{s})$ yields, on multiplication by $N$, the order of $G$, the number of elements in the class $C_q$. Finally the orthogonality relations (1) express the fact that the numbers $u_p{}^q = \sqrt{N_q/N} \cdot \chi_p{}^q$ are the elements of an $r \times r$ unitary matrix; so that

$$(2) \qquad \sum_{a=1}^{r} \chi_a{}^q \bar{\chi}_a{}^p = 0;\ q \neq p; \qquad \sum_{a=1}^{r} \chi_a{}^q \bar{\chi}_a{}^q = N/N_q.$$

Hence the equations $\phi_p(\boldsymbol{s}) = \frac{1}{N}\sum_q N_q \bar{\chi}_p{}^q s^q$ may be solved for the indeterminates $s^q$ the solution being

$$(3) \qquad s^j = \sum_a \chi_a{}^j \phi_a(\boldsymbol{s}).$$

Before passing to our subject proper, the symmetric group, it is necessary to say a few words concerning a basic theorem of Frobenius which enables us to derive from a characteristic of a subgroup $H$ of $G$ a characteristic of $G$ itself. Let $H$ be of order $M$ and denote by $h$ a typical element of $H$; the class of $H$ to which $h$ belongs is a subset, proper or not, of the class of $G$ to which $h$ belongs. But a class $C_j$ of $G$ may contain several classes of $H$ or none at all; we say that $H$ refines the classes of $G$. If $\Gamma$ is any representation of $G$ it induces a representation $\Gamma^-$ of $H$ where $\Gamma^-$ consists of those linear operators of $\Gamma$ which remain after the operators which correspond to elements of $G$ which are not in $H$ are rejected. If, in particular, $\Gamma$ is an irreducible representation of $G$ $\Gamma^-$ will be, in general, a *reducible* representation of $H$ (since it may be possible to find a proper, non-trivial subspace of the carrier space of $\Gamma$ which is invariant under all the operators of $\Gamma^-$ although the irreducibility of $\Gamma$ guarantees that no such subspace exists which is invariant under all the operators of $\Gamma$). If we have any class function $a(g)$ defined over $G$ it induces by the process of projection: $a^*(h) = a(h)$ a class function $a^*(h)$ defined over $H$. Since $a(g) = \sum_{a=1}^{r} (\boldsymbol{a} \cdot \boldsymbol{\chi}_a)\chi_a(g)$ we have $a^*(h) = \sum_{a=1}^{r} (\boldsymbol{a} \cdot \boldsymbol{\chi}_a)\chi_a(h)$ or, equivalently, $\boldsymbol{a}^* = \sum_{a=1}^{r} (\boldsymbol{a} \cdot \boldsymbol{\chi}_a)\boldsymbol{\chi}^*_a$. In particular when $\boldsymbol{a}$ is the indeterminate $\boldsymbol{s}$ whose components $s^q$ appeared in the definition of the simple characteristics of $G$ we have $\boldsymbol{s}^* = \sum_{a=1}^{r} \phi_a(\boldsymbol{s})\boldsymbol{\chi}^*_a$ where the numbers $\chi^*_p{}^j$ are the characters of a representation (in general reducible) of $H$ the index $j$ running over the classes

of $H$. If these number $t$ and if the characters of the irreducible representations of $H$ be denoted by $\xi_k{}^j$ $(k, j = 1, \cdots, t)$, we may write $\boldsymbol{\chi}^*{}_p = c_p{}^a \boldsymbol{\xi}_a$ where the coefficients $c_p{}^j$ are positive integers. The expression $(\boldsymbol{s}^* \cdot d^\beta \boldsymbol{\xi}_\beta) = d^\beta(\boldsymbol{s}^* \cdot \boldsymbol{\xi}_\beta)$ where the coefficients $d^j$ are integers, positive, negative or zero, is a generalised characteristic of $H$, it being clearly understood that the indeterminates $s^*(h)$ are conditioned by the fact that they are the same for all elements of $H$ which lie in the same class of $G$ (and not merely the same for all elements of $H$ which lie in the same class of $H$). On substituting for $s^*$ its expression given above this generalised characteristic of $H$ appears in the form

$$\sum_{a=1}^{r} \phi_a(\boldsymbol{s}) d^\beta(\boldsymbol{\chi}^*{}_a \cdot \boldsymbol{\xi}_\beta) = \sum_{a=1}^{r} \phi_a(\boldsymbol{s}) \sum_{\beta=1}^{t} (c_a{}^\beta d^\beta)$$

owing to the orthogonality relations amongst the characters of the irreducible representations of $H$. Since $\sum_{\beta=1}^{t} (c_a{}^\beta d^\beta)$ is an integer, positive, negative, or zero, it follows that *any* generalised characteristic of $H$ furnishes by the procedure outlined a generalised characteristic of $G$ (of which the components corresponding to classes of $G$ which contain no elements of $H$ are zero). As a trivial instance of this theorem let $G$ be the symmetric group on 2 letters and $H$ the identity element. The principal characteristic of $H$ is $s^1$ and this being a generalised characteristic of $G$ its components are $(2, 0)$ since $s^1 = \frac{1}{2}(2s^1 + 0 \cdot s^2)$. The generalised characteristic of $G$ obtained in this way contains the principal characteristic $\phi_1(\boldsymbol{s}) = \frac{1}{2}(s^1 + s^2)$ once since $c^1 = \frac{1}{2}(2.\,1 + 0.\,1) = 1$ and the remaining characteristic is $\frac{1}{2}(s^1 - s^2)$. This characteristic is simple since $\frac{1}{2}\{(1)^2 + (-1)^2\} = 1$ and, in addition, the coefficient of $s^1$ is positive. Thus the two simple characteristics of the symmetric group on two letters are $\phi_1(\boldsymbol{s}) = \frac{1}{2}(s^1 + s^2)$ and $\phi_2(\boldsymbol{s}) = \frac{1}{2}(s^1 - s^2)$ the corresponding characters being $(1, 1)$ and $(1, -1)$ respectively. As a less trivial example let $G$ be the symmetric group on 3 letters; $N = 6$, $N_1 = 1$, $N_2 = 3$, $N_3 = 2$ and let $H$ be the symmetric group on two of the three letters. The principal characteristic of $H$ is $\frac{1}{2}(s^1 + s^2)$ and writing this in the form $\frac{1}{6}(3s^1 + 3s^2 + 0 \cdot s^3)$ its components (when viewed as a generalised characteristic of $G$) are $(3, 1, 0)$. It contains the principal characteristic $\phi_1(\boldsymbol{s}) = \frac{1}{6}(s^1 + 3s^2 + 2s^3)$ of $G$ once since $c^1 = \frac{1}{6}(1.3.1 + 3.1.1 + 2.0.1) = 1$ and the remaining characteristic is $\frac{1}{6}(2s^1 + 0.s^2 - 2.s^3)$ with components $(2, 0, -1)$. This characteristic is simple since $\frac{1}{6}(2^2 + 0^2 + 2.(-1)^2) = 1$ and the coefficient of $s^1$ is positive. We have, then, the two simple characteristics $\phi_1(\boldsymbol{s}) = \frac{1}{6}(s^1 + 3s^2 + 2s^3)$; $\phi_2(\boldsymbol{s}) = \frac{1}{6}(2s^1 + 0.s^2 - 2s^3)$ of the symmetric group on 3 letters. Starting

with the characteristic $\frac{1}{2}(s^1 - s^2)$ of $H$ we find $c^1 = 0$, $c^2 = 1$ the remaining characteristic $\phi_3(s) = \frac{1}{6}(s^1 - 3s^2 + 2s^3)$ being simple. $\phi_1(s)$, $\phi_2(s)$, $\phi_3(s)$ are the three simple characteristics of the symmetric group on 3 letters the corresponding characters being $(1, 1, 1)$, $(2, 0, -1)$, and $(1, -1, 1)$ respectively.

We obtain a representation, in general reducible, of $G$ in the following manner. Any subgroup $H$, of order $M$, of $G$ has $d = N/M$ right cosets $H_1 = H$; $H_2 = Hg_2, \cdots, H_d = Hg_d$ and if we define $(H'_1, \cdots, H'_d)$ by the equation $H'_p = H_p g$ where $g$ is an arbitrary element of $G$ the symbols $H'$ constitute a permutation of the symbols $H$; i. e. $\boldsymbol{H}' = P(g)\boldsymbol{H}$ where $P(g)$ is a permutation matrix. The matrices $P(g)$ furnish a representation of $G$; if $g_q$ is any member of the class $C_q$ of $G$ $(q = 1, \cdots, r)$, the character $\chi^q$ associated with the class $C_q$ in this representation is the number of ones in the diagonal of the permutation matrix $P(g_q)$ i. e. the number of elements $g_p$ for which $H_p g_q = H_p$. In other words $\chi^q$ is the number of elements $g_p$ for which $g_p g_q g_p^{-1} \subset H$, or, since this number is the same for each $g_q \subset C_q$, $\chi^q =$ (number of times $g_p C_q g_p^{-1}$ lies in $H$) $\div N_q$. But $g_p C_q g_p^{-1} = C_q$ so that as $p$ runs from 1 to $d$ we obtain $C_q : d = N/M$ times. Hence $\chi^q = N/M$ times the number of elements of $H \subset C_q \div N_q$. This suffices to show that the generalised characteristic of $G$ obtained from the *principal* characteristic of $H$ by the method of the previous paragraph is really a compound (or simple) characteristic; in fact the characteristic of that representation (by permutation matrices) which is furnished by the cosets of $H$ in $G$. For the principal characteristic of $H$ is $1/M \sum_h s(h)$; written as a generalised characteristic of $G$ it appears as $1/N \sum_h Ns(h)/M$ so that its components $c^a \chi_a{}^q$ are $N/M$ times the number of elements of $H$ in $C_q \div N_q$. These being precisely the characters of the representation referred to, the theorem stated follows since two representations with the same characters or characteristic, are equivalent.

**2. The principal and alternating characteristics of the symmetric group. Construction of reducible representations.** Each permutation of the symmetric group on $n$ letters may be written in a unique manner as a product of cycles, no letter appearing in more than one cycle. Two permutations with the same cycle structure, i. e. containing the same number $\alpha_1$ of cycles on one letter (= unary cycles), the same number, $\alpha_2$ of cycles on two letters (= binary cycles), the same number, $\alpha_3$ of ternary cycles and so on, belong to the same class. For example if $n = 5$ and $P = (12)(345)$, $Q = (23)(154)$ the permutation $T = \begin{pmatrix} 12345 \\ 23154 \end{pmatrix} = (123)(45)$ transforms $P$ into $Q : Q = TPT^{-1}$. We

refer to the class with the cycle structure $(\alpha) = (\alpha_1, \cdots, \alpha_n)$ as the class $(\alpha)$ and observe that $\alpha_1 + 2\alpha_2 + \cdots + n\alpha_n = n$ so that the number of classes, and hence of irreducible representations, is the number of solutions of this equation in non-negative integers. Writing

$$\alpha_1 + \alpha_2 + \cdots + \alpha_n = \lambda_1 ; \; \alpha_2 + \cdots + \alpha_n = \lambda_2 ; \cdots \alpha_n = \lambda_n$$

it is clear that

$$\lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_n ; \quad \lambda_1 + \lambda_2 + \cdots + \lambda_n = n.$$

If $k$ is such that $\lambda_k > 0$, $\lambda_{k+1} = \lambda_{k+2} = \cdots = \lambda_n = 0$ we have

$$\lambda_1 + \lambda_2 + \cdots + \lambda_k = n ; \quad \lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_k > 0$$

so that the number of classes, or of irreducible representations, is the same as the number of partitions of $n$ into sums of positive integers. We shall indicate a partition by the symbol $(\lambda_1, \lambda_2, \cdots, \lambda_k)$ where the parts $\lambda_1, \cdots, \lambda_k$ are written in non-increasing order and shall use an obvious exponent notation for the sake of brevity when two or more parts are equal. E. g. $(3, 2^2, 1^3)$ denotes the partition $(3, 2, 2, 1, 1, 1)$ of $10$: $3 + 2 + 2 + 1 + 1 + 1 = 10$. Each partition is conveniently indicated by a diagram of horizontal rows of dots all beginning on the same vertical line; thus $(3, 2^2, 1^3)$ is indicated by the diagram

$$\begin{array}{lll} \cdot & \cdot & \cdot \\ \cdot & \cdot & \\ \cdot & \cdot & \\ \cdot & & \\ \cdot & & \\ \cdot & & \end{array}$$

By a simple interchange of the rows and columns of a diagram we obtain a second diagram (termed the *associate* of the original diagram) and hence a second partition of $n$ (termed the associate of the original partition). E. g., the associate of $(3, 2^2, 1^3)$ is $(6, 3, 1)$. When the associate is identical with the original the diagram (and partition) are termed *self-associated.* E. g., $(3, 2, 1)$ is a self-associated partition of 6. We shall see how to attach to each diagram, or partition of $n$, a uniquely determined irreducible representation of the symmetric group and then the representations attached to associated

diagrams, or partitions of $n$, are termed associated; a representation attached to a self-associated diagram or partition being termed self-associated. We shall denote the partition associated with $(\lambda)=(\lambda_1,\lambda_2,\cdots,\lambda_k)$ by $(\mu)=(\mu_1,\cdots,\mu_j)$ and it is an immediate consequence of the definition that $\mu_1=k$, $j=\lambda_1$; it is also clear that there are $\alpha_1$ ones, $\alpha_2$ twos, $\alpha_3$ threes etc. in the partition $(\mu)$ where $\alpha_1=(\lambda_1-\lambda_2)$; $\alpha_2=(\lambda_2-\lambda_3),\cdots,\alpha_n=\lambda_n$ or, equivalently, $\alpha_1+\alpha_2+\cdots+\alpha_n=\lambda_1$; $\alpha_2+\cdots+\alpha_n=\lambda_2$; $\cdots\alpha_n=\lambda_n$.

The number $N_{(\alpha)}$ of elements in the class $(\alpha)$ is readily found. If any such permutation is written down with the $\alpha_1$ unary cycles appearing first, the $\alpha_2$ binary cycles next, and so on, we obtain by mere permutation of the letters $n!$ permutations all in the class $(\alpha)$. But there are repetitions due to the fact that each cycle may begin, without changing it, with any one of its letters and to the fact that the $\alpha_r$ $r$-cycles may be permuted amongst themselves without affecting the permutation. Hence

$$N_{(\alpha)}=\frac{n!}{\prod_{p=1}^{p=n}p^{\alpha_p}\cdot\alpha_p!}=\frac{n!}{1^{\alpha_1}\cdot\alpha_1!\,2^{\alpha_2}\cdot\alpha_2!\cdots n^{\alpha_n}\cdot\alpha_n!}.$$

If $(s_1,\cdots,s_n)$ are $n$ indeterminates the expressions $s^{(\alpha)}=s_1^{\alpha_1}s_2^{\alpha_2}\cdots s_n^{\alpha_n}$ are class functions defined over the symmetric group and we use them in the definition of the simple characteristics of the symmetric group:

$$\phi_{(\lambda)}(\mathbf{s})=(\mathbf{s}\cdot\chi_{(\lambda)})=\frac{1}{n!}\sum_{(\alpha)}N_{(\alpha)}\bar{\chi}_{(\lambda)}{}^{(\alpha)}s^{(\alpha)}.$$

We shall denote the principal characteristic (i. e. the simple characteristic corresponding to the identity representation) by $q_n(\mathbf{s})$ so that

$$q_n(\mathbf{s})=\sum_{(\alpha)}\frac{1}{\alpha_1!\,\alpha_2!\cdots\alpha_n!}\left(\frac{s_1}{1}\right)^{\alpha_1}\left(\frac{s_2}{2}\right)^{\alpha_2}\cdots\left(\frac{s_n}{n}\right)^{\alpha_n}. \tag{4}$$

These polynomials $(n=1,2,\cdots)$ in the indeterminates $(s_1,s_2,\cdots)$ are the bricks with which will be built the characters of the irreducible representations of the symmetric group and we write out explicitly the first seven of them:

$$q_1(\mathbf{s})=s_1;\qquad q_2(\mathbf{s})=\tfrac{1}{2}(s_1^2+s_2);\qquad q_3(\mathbf{s})=\frac{1}{3!}(s_1^3+3s_1s_2+2s_3)$$

$$q_4(\mathbf{s})=\frac{1}{4!}\{s_1^4+6s_1^2s_2+8s_1s_3+3s_2^2+6s_4\}$$

$$q_5(\mathbf{s})=\frac{1}{5!}\{s_1^5+10s_1^3s_2+20s_1^2s_3+15s_1s_2^2+30s_1s_4+20s_2s_3+24s_5\}$$

$$q_6(s)=\frac{1}{6!}\{s_1^6+15s_1^4s_2+40s_1^3s_3+45s_1^2s_2^2+90s_1^2s_4+120s_1s_2s_3$$
$$+144s_1s_5+15s_2^3+90s_2s_4+40s_3^2+120s_6\}$$
$$q_7(s)=\frac{1}{7!}\{s_1^7+21s_1^5s_2+70s_1^4s_3+105s_1^3s_2^2+210s_1^3s_4+420s_1^2s_2s_3$$
$$+504s_1^2s_5+105s_1s_2^3+630s_1s_2s_4+280s_1s_3^2+840s_1s_6$$
$$+210s_2^2s_3+504s_2s_5+420s_3s_4+720s_7\}.$$

The terms are arranged so that $s_1^{m_1}s_2^{m_2}\cdots$ comes before $s_1^{n_1}s_2^{n_2}\cdots$ if the first non-vanishing number of the set $m_1-n_1$, $m_2-n_2,\cdots$ is positive. The polynomials $q_n(s)$ furnish at a glance the structure of the corresponding symmetric group. Thus from $q_6(s)$ we see that the $6!=720$ permutations of the symmetric group on 6 letters divide into 11 classes there being 45 elements, for example, in the class $\alpha=(2,2,0,0,0,0)$. It is clear from the defining formula (4) that the polynomials $q_n(s)$ satisfy the interconnecting relations

$$\frac{\partial q_n}{\partial s_1}=q_{n-1};\qquad \frac{\partial q_n}{\partial s_2}=\tfrac{1}{2}q_{n-2};\qquad \frac{\partial q_n}{\partial s_3}=\tfrac{1}{3}q_{n-3}\text{ and, generally,}$$

$$\frac{\partial q_n}{\partial s_p}=\frac{1}{p}\,q_{n-p},\qquad (p=1,2,\cdots,n)\tag{5}$$

where, to secure the universal validity of these formulae, we define $q_0(s)=1$; $q_{-1}(s)=q_{-2}(s)=\cdots=0$. We shall see that the relations (5) have the following significance; they enable us to construct, in a very simple manner, the characters of any class of the symmetric group on $n$ letters which contains at least one cycle on $p$ letters, $p=1,\cdots,n-1$, from the, supposed known, characters of the symmetric group on $(n-p)$ letters.

Since any cycle on $p$ letters may be written as the product of $p-1$ binary cycles (= transpositions): E. g., $(1234)=(12)(13)(14)$, the order of the factors being from left to right: every cycle on an even number of letters is an odd permutation and every cycle on an odd number of letters is an even one. Hence all permutations in a given class $(\alpha)$ are either even or odd and we may speak of even or odd classes; a class $(\alpha)$ being even when $\alpha_2+\alpha_4+\alpha_6+\cdots$ is even and odd when it is odd. Now the symmetric group on $n$ letters possesses, in addition to the identity representation, a second one-dimensional representation; namely that one which attaches to each even permutation the number 1 and to each odd permutation the number $-1$ (this being merely a sophisticated way of saying that the product of two even or odd permutations is even whilst the product of an even by an odd permutation is odd). This representa-

tion is known as the alternating representation and the corresponding simple characteristic is known as the alternating characteristic; we shall denote it by $\pi_n(s)$ so that

$$\pi_n(s) = \sum_{(\alpha)} \frac{(-1)^{\alpha_2+\alpha_4+\cdots}}{\alpha_1!\,\alpha_2!\cdots\alpha_n!}\left(\frac{s_1}{1}\right)^{\alpha_1}\left(\frac{s_2}{2}\right)^{\alpha_2}\cdots\left(\frac{s_n}{n}\right)^{\alpha_n}$$

implying

$$\pi_n(s_1, s_2, \cdots, s_n) = q_n(s_1, -s_2, s_3, -s_4, \cdots). \tag{6}$$

Before passing to the question of associating with each partition $(\lambda)$ of $n$ a (reducible) representation of the symmetric group on $n$ letters we find it convenient to remark that the characters $\chi_{(\lambda)}{}^{q}$ of the irreducible representations are all real (it appears in the sequel that they are all integers, positive, negative or zero, but this fact does not lie on the surface as does the fact of their reality). Indeed since the reciprocal of a cycle is the same cycle written in the reversed sense: E. g., $(1234)^{-1} = (1432)$: each class $(\alpha)$ contains the reciprocal of each of its permutations so that the character of any element is the same as the character of its reciprocal. But every representation of any *finite* group is equivalent to a representation by means of unitary matrices and, the reciprocal of a unitary matrix being its transposed conjugate, its trace is the conjugate complex of the trace of the original matrix. Hence the character $\chi(g^{-1})$ (for *any* representation of *any* finite group) is furnished by the relation $\chi(g^{-1}) = \bar{\chi}(g)$. For the *symmetric* group we have, in addition, the relation $\chi(g^{-1}) = \chi(g)$ and the two relations together imply $\bar{\chi}(g) = \chi(g)$ i. e. the reality of the characters of *any* representation of the symmetric group. We may, therefore, drop the conjugate complex sign in the explicit expressions for the simple characteristics and write

$$\phi_{(\lambda)}(s) = \sum_{(\alpha)} \frac{\chi_{(\lambda)}{}^{(\alpha)}}{\alpha_1!\,\alpha_2!\cdots\alpha_n!}\left(\frac{s_1}{1}\right)^{\alpha_1}\cdots\left(\frac{s_n}{n}\right)^{\alpha_n}. \tag{7}$$

In order to associate with each partition $(\lambda)$ of $n$ a (reducible) representation of the symmetric group on $n$ letters, we have merely to imagine the $n$ letters placed in compartments or boxes containing, respectively, $\lambda_1, \lambda_2, \cdots, \lambda_k$ letters and then to consider the subgroup $H$ of the symmetric group $G$ which consists of those permutations which do not send any letter out of its box. This subgroup is of order $M = \lambda_1!\,\lambda_2!\cdots\lambda_k!$ and a typical permutation of it is of the form $P = P_1P_2\cdots P_k$ where $P_j$ denotes a permutation on the letters of the $j$-th box $(j = 1, 2, \cdots, k)$. Since the various $P_j$ operate on distinct letters the order in which the factors $P_j$ are written is indifferent and we

agree to write them in the natural order $P_1P_2\cdots P_k$. If $(\alpha^j)=(\alpha_1^j,\cdots,\alpha_n^j)$ denotes the cycle structure of $P_j$ then the cycle structure of $P$ is furnished by the formula

$$\alpha_p=\sum_{j=1}^{k}\alpha_p^j; \qquad (p=1,2,\cdots,n).$$

Hence

$$s^{(\alpha)}=s_1^{\alpha_1}s_2^{\alpha_2}\cdots s_n^{\alpha_n}=\prod_{j=1}^{k}s^{(\alpha^j)}$$

and so the principal characteristic of $H$ is

$$\frac{1}{\lambda_1!\,\lambda_2!\cdots\lambda_k!}\prod_{j=1}^{k}\sum_j s^{(\alpha_j)}$$

where $\sum_j$ denotes summation over the $\lambda_j!$ permutations on the letters in the $j$-th box. On writing this principal characteristic of $H$ in the form

$$\prod_{j=1}^{k}\left(\frac{1}{\lambda_j!}\sum_j s^{(\alpha^j)}\right)$$

it appears as $q_{\lambda_1}(s)q_{\lambda_2}(s)\cdots q_{\lambda_k}(s)$. This product is, accordingly, a compound characteristic of the symmetric group $G$ on $n$ letters; namely, the characteristic of the representation of $G$ furnished by the permutations of the cosets of $H$ in $G$. Since the representation is by means of permutation matrices its characters are integers positive or zero and so the components of the characteristic (compound) $q_{\lambda_1}(s)\cdots q_{\lambda_k}(s)$ of $G$ are integers positive or zero. We shall denote by $\Delta(\lambda_1,\lambda_2,\cdots,\lambda_k)$ the representation (reducible) of the symmetric group on $n$ letters whose characteristic is $q_{\lambda_1}(s)\cdots q_{\lambda_k}(s)$.

$\Delta(\lambda_1,\cdots,\lambda_k)$ is sometimes referred to as a tensor representation for a reason that will be clear from the following examples:

1. If $k=2$, so that $n$ is partitioned into two parts, $\Delta(\lambda_1,\lambda_2)$ is of dimension $\binom{n}{\lambda_2}=\frac{n!}{\lambda_1!\lambda_2!}$ and the cosets of $H$ permute like the products of $n$ letters $(x_1,\cdots,x_n)$ $\lambda_2$ at a time. These products may be regarded as a basis in a carrier "tensor" space of $\binom{n}{\lambda_2}$ dimensions in which $\Delta(\lambda_1,\lambda_2)$ is presented by means of permutation matrices. Thus for $n=5$, $\lambda=(3,2)$ there are 10 products $x_1x_2,\cdots,x_4x_5$; the characters of $\Delta(3,2)$ are the number of these products which are left invariant by the permutations of the various classes and are at once seen to be $(10,4,1,2,0,1,0)$ which checks with the result

$$q_3(s)q_2(s)=\frac{1}{5!}\{10s_1^5+40s_1^3s_2+20s_1^2s_3+30s_1s_2^2+20s_2s_3\}$$

2. $\lambda_1=n-2$, $\lambda_2=1$, $\lambda_3=1$. The cosets permute like the products $x_1^2x_2$. For

$n=5$ $\Delta(3,1^2)$ is of dimension 20 and its characters are $(20,6,2,0,0,0,0)$ (the permutation $(12)$, for instance, leaving invariant the six products $x_3{}^2x_4$, $x_3x_4{}^2$, $x_4{}^2x_5$, $x_4x_5{}^2$, $x_5{}^2x_3$, $x_3x_5{}^2$); a result which checks with

$$q_3(s)q_1{}^2(s)=\frac{1}{5!}\{20s_1{}^5+60s_1{}^3s_2+40s_1{}^2s_3\}.$$

It is clear that the space spanned by the expressions $x_1x_2(x_1+x_2)$ etc. is an invariant subspace of the carrier space of $\Delta(n-2,1^2)$ as is also the space spanned by the expressions $x_1{}^2(x_2+x_3+\cdots+x_n)$ so that $\Delta(n-2,1^2)$ is in general reducible. Similarly the basic tensors for $\Delta(n-3,2,1)$ are $x_1{}^2(x_2x_3)$; for $\Delta(n-3,1^3)$ they are $x_1{}^3x_2{}^2x_3$; for $\Delta(n-4,3,1)$ they are $x_1{}^2(x_2x_3x_4)$; for $\Delta(n-4,2^2)$ they are $(x_1x_2)^2(x_3x_4)$; for $\Delta(n-4,2,1^2)$ they are $x_1{}^3x_2{}^2(x_3x_4)$ and so on. The attempt to solve one of the main problems of the present paper; namely, the analysis of the reducible representation $\Delta(\lambda_1,\cdots,\lambda_k)$ into its irreducible components, by the geometrical method of "tensor representations" soon becomes hopelessly complicated and we shall make no use of this geometrical viewpoint.

The subgroup $H$ whose elements are $P=P_1P_2\cdots P_k$ may be termed the direct product of the subgroups $G_1, G_2,\cdots, G_k$ where $G_j$ permutes only the letters in the $j$-th box, leaving all the other letters fixed $(j=1,\cdots,k)$; so that $G_j$ is of order $\lambda_j!$ We indicate the direct product relationship thus: $H=G_1\times G_2\times\cdots\times G_k$ and observe that if $\Gamma_j$ is any representation of $G_j$ $(j=1,\cdots,k)$, the Kronecker product $\Gamma_1\times\Gamma_2\times\cdots\times\Gamma_k$ is a representation of $H$ whose characters are the products of the corresponding characters of $\Gamma_1,\cdots,\Gamma_k$. If $\phi_j(s)$ is the characteristic of $G_j$ associated with $\Gamma_j$ the characteristic of $H$ associated with $\Gamma_1\times\Gamma_2\times\cdots\Gamma_k$ is, accordingly $\prod_{j=1}^{k}\phi_j(s)$ and this furnishes a compound characteristic of $G$; (when the representations $\Gamma_j$ $(j=1,2,\cdots,k)$, are all the identity representation $\phi_j(s)=q_{\lambda_j}(s)$ and we recover the compound characteristic $\prod_{j=1}^{k}q_{\lambda_j}(s)$ of $G$). We shall be particularly interested in the sequel in the case $k=2$; $\Gamma_1$ will be an irreducible representation of the symmetric group on $\lambda_1$ letters and $\Gamma_2$ an irreducible representation of the symmetric group on $\lambda_2$ letters. The compound characteristic of the symmetric group on $n=\lambda_1+\lambda_2$ letters obtained in the manner described above corresponds to a reducible representation of this group on $n$ letters which may be termed the direct product of $\Gamma_1$ and $\Gamma_2$ (and denoted $\Gamma_1\cdot\Gamma_2$). Our problem is the analysis of $\Gamma_1\cdot\Gamma_2$ into its irreducible components. The dimension of the direct product $\Gamma_1\cdot\Gamma_2$ is the product of the dimensions of $\Gamma_1$ and $\Gamma_2$ by $n!\div\lambda_1!\,\lambda_2!$, since its characteristic is $\phi_1(s)\phi_2(s)$ and the coefficient of the highest power of $s_1$ in this product is $\dfrac{1}{\lambda_1!}\,\dfrac{1}{\lambda_2!}$ times the product of the dimensions of $\Gamma_1$ and $\Gamma_2$.

**The formula of Frobenius for the simple characteristics of the symmetric group and its modification by Schur.** If we suppose the indeterminates $(s_1, \cdots, s_n)$ which occur in the expressions for the characteristics of the symmetric group to be the power sums of other indeterminates $(z_1, \cdots, z_n)$:

$$s_k = z_1^k + z_2^k + \cdots + z_n^k; \qquad (k = 1, \cdots, n)$$

the principal characteristic $q_m(s)$ of the symmetric group on $m \leq n$ letters becomes, when expressed in terms of the indeterminates $(z_1, \cdots, z_n)$, merely the complete homogeneous symmetric function $p_m(z)$ of degree $m$ in the $n$ variables $(z_1, \cdots, z_n)$. The first few of these functions are

$$p_0(z) = 1;\ p_1(z) = \Sigma z_1;\ p_2(z) = \Sigma z_1^2 + \Sigma z_1 z_2;\ p_3(z) = \Sigma z_1^3 + \Sigma z_1^2 z_2 + \Sigma z_1 z_2 z_3$$

and they are the coefficients of the development of the *generating* function $f(t) = \{(1 - z_1 t)(1 - z_2 t) \cdots (1 - z_n t)\}^{-1}$ in a power series in $t$:

$$f(t) = p_0(z) + p_1(z)t + p_2(z)t^2 + \cdots = \sum_0^\infty p_j(z)t^j.$$

But

$$\log f(t) = \sum_{p=1}^{n} (z_p t + \tfrac{1}{2} z_p^2 t^2 + \tfrac{1}{3} z_p^3 t^3 + \cdots) = s_1 t + \tfrac{1}{2} s_2 t^2 + \tfrac{1}{3} s_3 t^3 + \cdots$$

so that

$$\begin{aligned} f(t) &= e^{s_1 t}\, e^{\frac{1}{2} s_2 t^2}\, e^{\frac{1}{3} s_3 t^3} \cdots \\ &= \left\{ \sum_0^\infty \frac{1}{\alpha_1!}\left(\frac{s_1}{1}\right)^{\alpha_1} t^{\alpha_1} \right\} \left\{ \sum_0^\infty \frac{1}{\alpha_2!}\left(\frac{s_2}{2}\right)^{\alpha_2} t^{2\alpha_2} \right\} \cdots \\ &= \sum_{j=0}^\infty \left\{ \frac{1}{\alpha_1!} \cdot \frac{1}{\alpha_2!} \cdots \frac{1}{\alpha_j!} \left(\frac{s_1}{1}\right)^{\alpha_1} \left(\frac{s_2}{2}\right)^{\alpha_2} \cdots \left(\frac{s_j}{j}\right)^{\alpha_j} \right\} t^{\alpha_1 + 2\alpha_2 + \cdots + j\alpha_j} \end{aligned}$$

and this implies $p_j(z) = q_j(s)$ $(j = 0, 1, 2, \cdots, n)$. The homogeneous products $p_j(z) = \Sigma(z_1^{t_1} z_2^{t_2} \cdots z_n^{t_n})$, $t_1 + t_2 + \cdots + t_n = j$ are intimately connected, in a reciprocal manner, with the elementary symmetric functions:

$$\sigma_0(z) = 1;\ \ \sigma_1(z) = \Sigma z_1;\ \ \sigma_2(z) = \Sigma z_1 z_2; \cdots\ \ \sigma_n(z) = z_1 z_2 \cdots z_n$$

either set being expressible as polynomials with integral coefficients in the other set. In fact the generating function for the elementary symmetric functions $\sigma_k(z)$ is

$$g(t) = \prod_{p=1}^{n} (1 + z_p t) = \sum_0^\infty \sigma_j(z) t^j$$

and on taking logarithms we find

$$\log g(t) = s_1 t - \tfrac{1}{2} s_2 t^2 + \tfrac{1}{3} s_3 t^3 - \cdots$$

so that $\sigma_j(z) = q_j(s_1, -s_2, s_3, -s_4, \cdots) = \pi_j(s)$. In other words the alternating characteristic of the symmetric group on $j \leq n$ letters becomes, when expressed in terms of the indeterminates $(z_1, \cdots, z_n)$, simply the elementary symmetric function $\sigma_j(z)$. The two generating functions $f(t)$ and $g(t)$ are such that $g(t) = \{f(-t)\}^{-1}$ and hence $\{\sum_0^\infty \sigma_j t^j\}\{\sum_0^\infty (-1)^k p_k t^k\} = 1$ and this yields the series of relations

$$\sigma_0 p_0 = 1; \quad \sigma_0 p_1 - \sigma_1 p_0 = 0; \quad \sigma_0 p_2 - \sigma_1 p_1 + \sigma_2 p_0 = 0; \cdots.$$

These may be expressed by the statement that the two matrices

$$P_j = \begin{pmatrix} p_0 & p_1 & \cdots & p_{j-1} \\ & p_0 & \cdots & p_{j-2} \\ & & \cdots & \\ & & & p_0 \end{pmatrix} \quad \text{and} \quad \Sigma_j = \begin{pmatrix} \sigma_0 & -\sigma_1 & \sigma_2 & \cdots \\ & \sigma_0 & -\sigma_1 & \cdots \\ & & & \cdots \\ & & & \sigma_0 \end{pmatrix}; \qquad (j = 1, 2, \cdots)$$

are reciprocal (the elements below the diagonal in each matrix being zero). Since $p_0 = 1 = \sigma_0$ the determinant of each matrix is unity and so each element of either is a cofactor of the other; thus

$$\sigma_1 = p_1; \quad \sigma_2 = \begin{vmatrix} p_1 & p_2 \\ p_0 & p_1 \end{vmatrix}; \quad \sigma_3 = \begin{vmatrix} p_1 & p_2 & p_3 \\ p_0 & p_1 & p_2 \\ 0 & p_0 & p_1 \end{vmatrix}$$

and so in general. $\sigma_r$ is an $r$ rowed determinant whose diagonal elements are all $p_1$ the non-diagonal elements being obtained by increasing the suffix carried by $p$ methodically by one as we move from each column to its right-hand neighbor and decreasing this suffix by one as we move from each column to its left-hand neighbor (it being understood that $p_{-1} = p_{-2} = \cdots = 0$). The result of this calculation needed for our immediate purpose lies on the surface: the symmetric functions $p_j(z)$ may be used instead of the elementary symmetric functions $\sigma_k(z)$ as a basis for symmetric functions. More particularly any symmetric polynomial in $z$ with integral coefficients may be expressed as a linear combination of products of the functions $p_k$ with integral coefficients; and if the polynomial is homogeneous of degree $n$ the products entering the linear combination are of the type $p_{\lambda_1}(z) p_{\lambda_2}(z) \cdots p_{\lambda_k}(z)$ where

$\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$. Since $p_j(z) = q_j(s)$ this furnishes the basic result: any homogeneous symmetric polynomial of degree $n$ in the variables $(z_1, \cdots, z_n)$, with integral coefficients, is, when expressed in terms of the variables $(s_1, \cdots, s_n)$ a generalised characteristic of the symmetric group on $n$ letters. Frobenius' essential contribution to the theory was the discovery of those particular symmetric functions of $z$ which yield the *simple* characteristics; and to Schur we owe the recognition of the importance of a very elegant and useful expression of them (due to Jacobi) as determinants whose elements are the functions $p_j(z) = q_j(s)$, $j \leq n$.

The expression just given for $\sigma_r(z)$ as a determinant whose elements are members of the set $p_j(z)$ is merely a special case of a general reciprocal relationship between determinants whose elements are members of the set $\sigma_j(z) = \pi_j(s)$ and determinants whose elements are members of the set $p_j(z) = q_j(s)$; which merely reflects the fact that the matrices $P_j$ and $\Sigma_j$ are reciprocal. We shall need a special case of this relationship and it is convenient to derive it here. Let $(\lambda) = (\lambda_1, \cdots, \lambda_k)$ be a partition of $n$ and consider the determinant

$$\{\lambda\} = \begin{vmatrix} p_{\lambda_1} & p_{\lambda_1+1} & \cdots & p_{\lambda_1+k-1} \\ p_{\lambda_2-1} & p_{\lambda_2} & \cdots & p_{\lambda_2+k-2} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ p_{\lambda_k-k+1} & & \cdots & p_{\lambda_k} \end{vmatrix}.$$

This determinant is a certain $k$ rowed minor of the matrix $P_{\lambda_1+k}$; in fact the one obtained by erasing the first $\lambda_1$ columns and retaining the 1st, the $\lambda_1 - \lambda_2 + 2)$-nd, the $(\lambda_1 - \lambda_3 + 3)$-rd, $\cdots$ and the $(\lambda_1 - \lambda_k + k)$-th rows. Save for a question of sign, into which it is profitless to go since it can be settled in a trivial manner later, $\{\lambda\}$ is, therefore, equal to that minor of the reciprocal matrix $\Sigma_{\lambda_1+k}$ which is obtained by *keeping* the first $\lambda_1$ *rows* and *omitting* the 1-st, the $(\lambda_1 - \lambda_2 - 2)$-nd, the $(\lambda_1 - \lambda_3 + 3)$-rd $\cdots$ and the $(\lambda_1 - \lambda_k + k)$-th *columns*. Since $\lambda_k > 0$, the last column of $\Sigma_{\lambda_1+k}$ is kept and the suffix attached to the $\sigma$ in the lower right-hand corner is $k$ (for the minor has $\lambda_1$ rows and the suffix attached to the $\sigma$ in the upper right-hand corner is $\lambda_1 + k - 1$ whilst the suffixes diminish methodically by one as we step from each row to its neighbor below). Counting from the last column the first column omitted is the $(\lambda_k + 1)$-st; and the suffixes of the last $\lambda_k$ diagonal elements of the minor of $\Sigma_{\lambda_1+k}$ in question all equal $k$; since the second column omitted is the $(\lambda_{k-1} + 2)$-nd, counting from the last, and so on, the next diagonal suffix, counting upwards to the left, is less than $k$ by the number

of $\lambda$'s that equal $\lambda_k$. Continuing in this way we see that the diagonal suffixes of the minor of $\Sigma_{\lambda_1+k}$ constitute the partition $(\mu) = (\mu_1, \mu_2, \cdots, \mu_{\lambda_1})$ of $n$ which is associated with the partition $(\lambda)$ of $n$. For instance if $n = 4$ and $(\lambda) = (2, 1^2)$ so that $(\mu) = (3, 1)$ we have proved that

$$\{2, 1^2\} = \begin{vmatrix} p_2 & p_3 & p_4 \\ p_0 & p_1 & p_2 \\ 0 & p_0 & p_1 \end{vmatrix} = \pm \begin{vmatrix} -\sigma_1 & \sigma_4 \\ \sigma_0 & -\sigma_3 \end{vmatrix}.$$

The negative signs may be removed from the $\sigma$'s carrying odd labels by changing the signs of all columns having a $\sigma$ with an odd suffix at the bottom and following this by a change of sign of all rows having a $\sigma$ with an even suffix at the end. On reflecting the $\sigma$ minor about its secondary diagonal (an operation which does not affect the value of the determinant) we find

$$\{\lambda\} \equiv \begin{vmatrix} p_{\lambda_1} & \cdot & \cdot & \cdot & \cdot \\ \cdot & p_{\lambda_2} & & & \cdot \\ \cdot & \cdot & \cdot & \cdot & p_{\lambda_k} \end{vmatrix} = \pm \begin{vmatrix} \sigma_{\mu_1} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \sigma_{\mu_2} & & & \cdot \\ \cdot & \cdot & \cdot & \cdot & \sigma_{\mu_j} \end{vmatrix}; \qquad (\mu_1 = k,\ j = \lambda_1)$$

where the non-diagonal elements of each determinant are filled in by increasing methodically the suffixes by one as we move to the right (and diminishing them by one as we move to the left); it being understood that a $p$ or $\sigma$ carrying a negative suffix is to be replaced by zero and that the partitions $(\lambda)$ and $(\mu)$ of $n$ are associated. That the undetermined sign is $+$, rather than $-$, is immediately evident when we recall that $p_j(z) = q_j(s)$

$$\sigma_j(z) = \pi_j(s) = q_j(s_1, -s_2, s_3, -s_4, \cdots).$$

On setting $s_1 = 1$, $s_2 = s_3 = \cdots = s_n = 0$, $p_j(z) = q_j(s)$ takes the value $1/j!$ as also does $\sigma_j(z) = \pi_j(s)$. And on writing

$$\lambda_1 + (k-1) = l_1; \ \lambda_2 + (k-2) = l_2; \ \cdots \lambda_k = l_k$$

it is clear that the determinant $\{\lambda\}$ becomes the quotient by $l_1!\, l_2! \cdots l_k!$ of a $k$ rowed determinant of which the element in the $j$-th row and $p$-th column is $l_j(l_j - 1) \cdots (l_j + p + 1 - k)$ there being $k - p$ factors (the elements in the $k$-th column being all unity). Since the element in the $j$-th row and the $p$-th column is a polynomial in $l_j$ of degree $k - p$ (the coefficients of the polynomial being independent of the row number $j$ and the coefficient of the highest power being unity) it is at once clear, on subtracting from each column

an appropriate linear combination of the succeeding columns, that the determinant is equivalent to the Vandermonde determinant whose $j$-th row is $(l_j^{k-1}, l_j^{k-2}, \cdots, l_j, 1)$. Its value is, therefore

$$\Delta(l) = \prod_{p<q} (l_p - l_q).$$

Since the partition $(\lambda)$ of $n$ was arranged in non-increasing order: $\lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_k > 0$ the numbers $(l)$ are arranged in *descending* order: $l_1 > l_2 > \cdots > l_k > 0$ and so $\Delta(l) > 0$. Hence when $s_1 = 1, s_2 = s_3 = \cdots s_n = 0$ both the determinants

$$\begin{vmatrix} p_{\lambda_1} & \cdots & \cdot \\ \cdot & p_{\lambda_2} & \cdot \\ \cdot & \cdots & p_{\lambda_k} \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} \sigma_{\mu_1} & \cdots & \cdot \\ \cdot & \sigma_{\mu_2} & \cdot \\ \cdot & \cdots & \sigma_{\mu_j} \end{vmatrix}$$

are positive; they must accordingly be equal and not one the negative of the other. Moreover if we set $\mu_1 + (j-1) = m_1, \cdots, \mu_j = m_j$ the two numbers $\Delta(l) \div l_1!\, l_2! \cdots l_k!$ and $\Delta(m) \div m_1!\, m_2! \cdots m_j!$ are equal. Finally we remark that the theorem of the present paragraph may be stated in the following convenient manner. Denoting by $\{\lambda\}^*$ the result of changing the signs of $s_2, s_4, \cdots$ in $\{\lambda\}$ then

$$\{\lambda\} = \{\mu\}^*$$

where $\{\lambda\}$ is the determinant

$$\begin{vmatrix} q_{\lambda_1}(\boldsymbol{s}) & \cdots & \cdot \\ \cdot & q_{\lambda_2}(\boldsymbol{s}) & \cdot \\ \cdot & \cdots & q_{\lambda_k}(\boldsymbol{s}) \end{vmatrix}$$

and $(\mu)$ is the partition of $n$ associated with $(\lambda)$.

We now proceed to the determination of the *simple* characteristics of the symmetric group on $n$ letters (remembering that *any* homogeneous symmetric polynomial of degree $n$ in the $n$ indeterminates $(z_1, \cdots, z_n)$ yields, when expressed in terms of their power sums $(s_1, \cdots, s_n)$ a *generalised* characteristic of this group). What is needed is a criterion which will decide whether or not a set of generalised characteristics are simple, and this is provided as follows. Let $\boldsymbol{\chi}_p$ $(p = 1, \cdots, r)$, denote the characters of the $r$ non-equivalent irreducible representations of the symmetric group on $n$ letters ($r$ being the number of partitions of $n$) so that $\phi_p(\boldsymbol{s}) = (\boldsymbol{s} \cdot \boldsymbol{\chi}_p)$ are the $r$ simple characteristics. Let $\boldsymbol{t}$ be a second indeterminate and form the products

2

$\phi_p(\boldsymbol{s}) \cdot \phi_p(\boldsymbol{t})$ and then sum these products as $p$ runs over the values $1, 2, \cdots, r$. We have

$$\phi_p(\boldsymbol{s}) = \frac{1}{n!} \sum_P \chi_p(P) s_1^{\alpha_1} \cdots s_n^{\alpha_n}$$

where the summation is over all elements $P$ of the symmetric group $((\alpha) = (\alpha_1, \cdots, \alpha_n)$ indicating the class to which $P$ belongs). Similarly

$$\phi_p(\boldsymbol{t}) = \frac{1}{n!} \sum_Q \chi_p(Q) t_1^{\beta_1} \cdots t_n^{\beta_n}$$

where $(\beta_1, \cdots, \beta_n)$ indicates the class to which $Q$ belongs. On forming the product and summing with respect to $p$ we have a triple summation; namely with respect to $P$, with respect to $Q$, and with respect to $p$. Performing first the summation with respect to $p$ we obtain zero unless $Q$ belongs to the same class as $P$ (owing to the orthogonality relations amongst the simple characters). For a fixed $P$ there are $N_{(\alpha)}$ choices of $Q$, namely all the $Q$'s in the same class as $P$, and summation with respect to $Q$ gives $\sum_p N_{(\alpha)} \chi_p^{(\alpha)} \chi_p^{(\alpha)} = n!$ There remains only the single summation with respect to $P$ and we find

$$\sum_p \phi_p(\boldsymbol{s}) \phi_p(\boldsymbol{t}) = \frac{1}{n!} \sum_P (s_1 t_1)^{\alpha_1} \cdots (s_n t_n)^{\alpha_n} = q_n(\boldsymbol{u})$$

where $\boldsymbol{u} = \boldsymbol{st}$ in the sense that $u_1 = s_1 t_1$, $u_2 = s_2 t_2, \cdots, u_n = s_n t_n$. The real force of this result lies in the fact that its converse is true in the following sense: suppose we have $r$ generalised characteristics $F_p(\boldsymbol{s})$ $(p = 1, \cdots, r)$, possessing the property that $\sum_{p=1}^{r} F_p(\boldsymbol{s}) F_p(\boldsymbol{t}) = q_n(\boldsymbol{st})$; then each of these characteristics is either a simple characteristic or the negative of one. In fact $F_p(\boldsymbol{s}) = c_p{}^{a} \phi_a(\boldsymbol{s})$, where the coefficients $c_p{}^{q}$ are integers, positive, negative or zero and so

$$\sum_{p=1}^{r} \phi_p(\boldsymbol{s}) \phi_p(\boldsymbol{t}) = q_n(\boldsymbol{st}) = \sum_{p=1}^{r} F_p(\boldsymbol{s}) F_p(\boldsymbol{t}) = \sum_{p=1}^{r} c_p{}^{a} c_p{}^{\beta} \phi_a(\boldsymbol{s}) \phi_\beta(\boldsymbol{t}).$$

Now the $r$ simple characteristics are linearly independent; for a hypothecated relation $c^a \phi_a(\boldsymbol{s}) = 0$ would imply $c^a \chi_a = 0$ and this would imply $c^p = 0$ $(p = 1, \cdots, r)$, owing to the orthogonality relations (1). Equating then, the coefficients of $\phi_q(\boldsymbol{s})$ on both sides of the equation just written, we obtain

$$\phi_q(\boldsymbol{t}) = \sum_{p=1}^{r} c_p{}^q c_p{}^\beta \phi_\beta(\boldsymbol{t}) ; \qquad (q = 1, 2, \cdots, r)$$

and this implies, again on account of the linear independence of the simple characteristics,

$$\sum_{p=1}^{r} c_p{}^q c_p{}^j = 0 ; \; j \neq q ; \qquad \sum_{p=1}^{r} (c_p{}^q)^2 = 1.$$

Since the numbers $c_p{}^q$ are integers it follows from the second of these two equations that, for a fixed $q$, all $c_p{}^q$ vanish save one which is $\pm 1$; and then the remaining equations show that for a fixed $p$ all $c_p{}^q$ vanish save one which is $\pm 1$. In other words the generalised characteristics $F_p(\boldsymbol{s})$ are merely a rearrangement of the simple characteristics $\phi_p(\boldsymbol{s})$ followed, possibly, by a change of sign of some of them.

Let $(v_1, \cdots, v_n)$ be $n$ integers, positive or zero, no two of which are equal, supposed arranged in descending order of magnitude: $v_1 > v_2 > \cdots > v_n$, and denote by $A(v_1, \cdots, v_n)$ the $n$ rowed determinant of which the elements in the $j$-th row are the $v^j$-th powers of the indeterminates $(z_1, \cdots, z_n)$ $(j = 1, \cdots, n)$. When $v_1 = n - 1, v_2 = n - 2, \cdots, v_n = 0, A(v_1, \cdots, v_n)$ is the Vandermonde determinant whose value is the difference product $\Delta = \Delta(\boldsymbol{z}) = \prod_{j<k} (z_j - z_k)$. It is clear that $A(v_1, \cdots, v_n)$ contains $\Delta$ as a factor and since both $A(v_1, \cdots, v_n)$ and $\Delta$ are alternating functions of $\boldsymbol{z}$ the quotient is symmetric and it is at once seen to be a polynomial of degree $(v_1 + v_2 + \cdots + v_n) - (\overline{n-1} + \overline{n-2} + 1 + 0)$ with integral coefficients. If, then, $(\lambda) = (\lambda_1, \cdots, \lambda_n)$ is a partition of $n$ and we write

$$v_1 = \lambda_1 + (n-1), \qquad v_2 = \lambda_2 + (n-2), \cdots, v_n = \lambda_n$$

the quotient $A(v_1, \cdots, v_n) \div \Delta$ is a symmetric polynomial of degree $n$, with integral coefficients, in the indeterminates $(z_1, \cdots, z_n)$; it furnishes, therefore, when expressed in terms of the power sums $(s_1, \cdots, s_n)$, a generalised characteristic of the symmetric group on $n$ letters and the basic result of Frobenius is to the effect that the characteristics obtained in this way are *simple*. Let us denote the quotient $A(v_1, \cdots, v_n) \div \Delta$ by $\{\lambda\}$; then in order to derive the result of Frobenius we have first to show that $\sum_{(\lambda)} \{\lambda\}(\boldsymbol{s})\{\lambda\}(\boldsymbol{t}) = q_n(\boldsymbol{st})$ and then that the coefficient of $s_1$ in $\{\lambda\}(\boldsymbol{s})$ is positive. We first remark that the relations

$$s_j = z_1{}^j + z_2{}^j + \cdots + z_n{}^j ; \qquad t_j = y_1{}^j + y_2{}^j + \cdots + y_n{}^j$$

imply $s_j t_j = \Sigma(z_p y_q)^j$; $(p = 1, \cdots, n;\ q = 1, \cdots, n)$. Hence if we denote the $n^2$ quantities $z_p y_q$ by $\boldsymbol{zy}$ the relations $\boldsymbol{s} \to \boldsymbol{z}$, $\boldsymbol{t} \to \boldsymbol{y}$ imply $\boldsymbol{st} \to \boldsymbol{zy}$ and, in particular, $q_n(\boldsymbol{st}) = p_n(\boldsymbol{zy})$; so that the first part of our problem may be re-phrased as follows: we must show that $\sum_{(\lambda)} \{\lambda\}(\boldsymbol{z})\{\lambda\}(\boldsymbol{y}) = p_n(\boldsymbol{zy})$ the summation being over the $r$ partitions of $n$.

To do this we first consider a determinant of order $n$ of which the element in the $i$-th row and $j$-th column is $(a_i + b_j)^{-1}$. On subtracting the first *column* from each of the others and removing the common factors

$$(b_1 - b_2)(b_1 - b_3) \cdots (b_1 - b_n) \div (b_1 + a_1)(b_1 + a_2) \cdots (b_1 + a_n)$$

we obtain a determinant of which the elements in the $i$-th row are 1, $(a_i + b_2)^{-1}, \cdots, (a_i + b_n)^{-1}$. On subtracting the first *row* of this determinant from each of the others and removing the common factors

$$(a_1 - a_2)(a_1 - a_3) \cdots (a_1 - a_n) \div (a_1 + b_2)(a_1 + b_3) \cdots (a_1 + b_n)$$

we obtain a determinant of order $n - 1$ of which the element in the $i$-th row and $j$-th column is again $(a_i + b_j)^{-1}$ where now $i$, $j$ run from 2 to $n$ instead of from 1 to $n$ as before. It follows at once that the $n$-th order determinant, of which the element in the $i$-th row and $j$-th column is $(a_i + b_j)^{-1}$, has the value $\Delta(\boldsymbol{a})\Delta(\boldsymbol{b}) \div \Pi(a_i + b_j)$; $(i = 1, \cdots, n,\ j = 1, 2, \cdots, n)$, where $\Delta(\boldsymbol{a})$ denotes the difference product $(a_1 - a_2) \cdots (a_{n-1} - a_n)$ (a result due to Cauchy). On writing $a_i = \alpha_i^{-1}$, $b_j = -\beta_j$ this result of Cauchy appears in the following equivalent form: the determinant of order $n$ of which the element in the $i$-th row and $j$-th column is $(1 - \alpha_i\beta_j)^{-1}$ has the value

$$\Delta(\boldsymbol{\alpha})\Delta(\boldsymbol{\beta}) \div \Pi(1 - \alpha_i\beta_j).$$

But if $\boldsymbol{A}$ denotes the $n \times \infty$ matrix of which the elements in the $i$-th *row* are $(1, \alpha_i, \alpha_i^2, \cdots)$ and $\boldsymbol{B}$ the $\infty \times n$ matrix of which the elements in the $j$-th *column* are $(1, \beta_j, \beta_j^2, \cdots)$ the product $\boldsymbol{A} \cdot \boldsymbol{B}$ is an $n \times n$ matrix of which the element in the $i$-th row and $j$-th column is $1 + \alpha_i\beta_j + \alpha_i^2\beta_j^2 + \cdots$ or $(1 - \alpha_i\beta_j)^{-1}$. The determinant of the product $\boldsymbol{AB}$ may be found by selecting *any* $n$-th order matrix from $\boldsymbol{A}$, multiplying its determinant by the determinant of the corresponding matrix from $\boldsymbol{B}$, and adding all products so obtained; that the number of products is infinite need cause no concern since $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are indeterminates and we may regard them so chosen that the components $\alpha_i, \beta_j$

are all $< 1$ in numerical magnitude so that the infinite series which appear are all absolutely convergent. All determinants of order $n$ selected from the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are of the type $A(p_1, \cdots, p_n)$ where we may, without lack of generality, agree that $p_1 > p_2 > \cdots > p_n > 0$. Hence we have

$$\sum_{(p)} A(p_1, \cdots, p_n)(\boldsymbol{\alpha}) . A(p_1, \cdots, p_n)(\boldsymbol{\beta}) = \Delta(\boldsymbol{\alpha})\Delta(\boldsymbol{\beta}) \div \Pi(1-\alpha_i\beta_j).$$

On setting $\boldsymbol{\beta} = \boldsymbol{\delta} t$ i. e. $\beta_1 = \delta_1 t$, $\beta_2 = \delta_2 t, \cdots, \beta_n = \delta_n t$, where $t$ is an indeterminate, we have $A(p_1 \cdots p_n)(\boldsymbol{\beta}) = A(p_1, \cdots, p_n)(\boldsymbol{\delta})\, t^{p_1+p_2+\cdots+p_n}$,

$$\Delta(\boldsymbol{\beta}) = \Delta(\boldsymbol{\delta})\, t^{(n-1)+(n-2)+\cdots+1+0}$$

and on writing

$$p_1 - (n-1) = \lambda_1;\ p_2 - (n-2) = \lambda_2;\ \cdots;\ p_n = \lambda_n$$

we find

$$\sum_{(\lambda)} \{\lambda\}(\boldsymbol{\alpha})\{\lambda\}(\boldsymbol{\delta})\, t^{\lambda_1+\cdots+\lambda_n} = \{\Pi(1-\alpha_i\delta_j t)\}^{-1}$$

$$= \sum_{0}^{\infty} p_n(\boldsymbol{\alpha\delta})\, t^n.$$

On equating coefficients of $t^n$ we obtain

$$\sum_{(\lambda)} \{\lambda\}(\boldsymbol{\alpha})\{\lambda\}(\boldsymbol{\delta}) = p_n(\boldsymbol{\alpha\delta})$$

where the summation is over all partitions $(\lambda)$ of $n$. This proves that the symmetric polynomials $\{\lambda\}(\boldsymbol{z})$, furnish, when expressed in terms of the power sums $(s_1, \cdots, s_n)$, either simple characteristics or the negatives of these; all simple characteristics being obtained in this way. To show that we have actually the simple characteristics, and not the negatives of any of them, we must show that the coefficient of $s_1{}^n$ in $\{\lambda\}(\boldsymbol{z}) > 0$. To do this we shall first derive Jacobi's expression for $\{\lambda\}(\boldsymbol{z})$ as a determinant whose elements are members of the set $p_j(\boldsymbol{z}) = q_j(\boldsymbol{s})$, $j \leq n$. Before doing this we remark that Frobenius stated his result in a slightly different form. From

$$\phi_{(\lambda)}(\boldsymbol{s}) = A(v_1, \cdots, v_n) \div \Delta(\boldsymbol{z})$$

and (3) we have

$$s^{(a)} = \sum_{(\lambda)} \chi_{(\lambda)}{}^{(a)} A(v_1, \cdots, v_n) \div \Delta(\boldsymbol{z})$$

so that $\chi_{(\lambda)}{}^{(a)}$ is the coefficient of $A(v_1, v_2, \cdots, v_n)$ in the development of

$$\Delta(\boldsymbol{z}) s^{(a)} = \{\prod_{p<q} (z_p - z_q)\} s_1{}^{a_1} s_2{}^{a_2} \cdots s_n{}^{a_n}.$$

To obtain Jacobi's expression it is necessary to point out some trivially evident properties of the homogeneous products $p_j(z_1, \cdots, z_n)$. The generating function for these products was $\{(1-z_1t)\cdots(1-z_nt)\}^{-1}$:

$$\{(1-z_1t)\cdots(1-z_nt)\}^{-1}=\sum_0^\infty p_j(z_1,z_2,\cdots,z_n)t^n.$$

It follows, on multiplication by $(1-z_nt)$, that

$$\sum_0^\infty p_j(z_1,z_2,\cdots,z_{n-1})t^j=(1-z_nt)\sum_0^\infty p_j(z_1,z_2,\cdots,z_n)t^j$$

so that $p_j(z_1,\cdots,z_{n-1})=p_j(z_1,\cdots,z_n)-z_np_{j-1}(z_1,\cdots,z_n)$ or, equivalently, $p_j(z_1,\cdots,z_n)=z_np_{j-1}(z_1,\cdots,z_n)+p_j(z_1,\cdots,z_{n-1})$. Applying this "reduction formula" to both terms on the right-hand side we obtain

$$\begin{aligned}p_j(z_1,z_2,\cdots,z_n)&=z_n\{z_np_{j-2}(z_1,\cdots,z_n)+p_{j-1}(z_1,\cdots,z_{n-1})\}\\&\qquad+z_{n-1}p_{j-1}(z_1,\cdots,z_{n-1})+p_j(z_1,\cdots,z_{n-2})\\&=z_n{}^2p_{j-2}(z_1,\cdots,z_n)+(z_n+z_{n-1})p_{j-1}(z_1,\cdots,z_{n-1})+p_j(z_1,\cdots,z_{n-2})\end{aligned}$$

a relation which may be written in the form

$$\begin{aligned}p_j(z_1,\cdots,z_n)&=p_2(z_n)p_{j-2}(z_1,\cdots,z_n)\\&\qquad+p_1(z_n,z_{n-1})p_{j-1}(z_1,\cdots,z_{n-1})+p_j(z_1,\cdots,z_{n-2})\end{aligned}$$

which suggests the relation

$$\begin{aligned}p_j(z_1,\cdots,z_n)&=p_m(z_n)p_{j-m}(z_1,\cdots,z_n)\\ \text{(8)}\qquad&\qquad+p_{m-1}(z_n,z_{n-1})p_{j-m+1}(z_1,\cdots,z_{n-1})+\cdots;\quad(m=1,2,\cdots).\end{aligned}$$

That this relation actually does hold is readily proved by induction; for assuming its validity for a stated value of $m$ its validity for $m+1$ follows at once. Thus

$$\begin{aligned}p_j(z_1,\cdots,z_n)&=p_m(z_n)\{z_np_{j-m-1}(z_1,\cdots,z_n)+p_{j-m}(z_1,\cdots,z_{n-1})\}\\&\qquad+p_{m-1}(z_n,z_{n-1})\{z_{n-1}p_{j-m}(z_1,\cdots,z_{n-1})+p_{j-m+1}(z_1,\cdots,z_{n-2})\}+\cdots\\&=p_{m+1}(z_n)p_{j-m-1}(z_1,\cdots,z_n)+p_m(z_n,z_{n-1})p_{j-m}(z_1,\cdots,z_{n-1})+\cdots.\end{aligned}$$

Since the relation (8) is true for $m=1$ it is true for every positive integer; it being always understood that a $p$ with a negative label is assigned the value zero. It is also understood that all $p_q(z_1,\cdots,z_s)$ are assigned the value zero when $s<1$.

We need one other property of the homogeneous products $p_j(z_1,\cdots,z_n)$. Writing the basic recurrence relation $p_j(z_1,\cdots,z_n)=z_np_{j-1}(z_1,\cdots,z_n)+p_j(z_1,\cdots,z_{n-1})$ in the equivalent form

$$p_{j-1}(z_1,\cdots,z_n)=p_{j-1}(z_1,\cdots,z_{n+1})-z_{n+1}p_{j-2}(z_1,\cdots,z_{n+1})$$

we find

$$p_j(z_1,\cdots,z_n)=z_n\{p_{j-1}(z_1,\cdots,z_{n+1})-z_{n+1}p_{j-2}(z_1,\cdots,z_{n+1})\}+p_j(z_1,\cdots,z_{n-1})$$

and, on interchanging $z_n$ and $z_{n+1}$ and subtracting,

$$\begin{aligned}\text{(9)}\quad p_{j-1}&(z_1,\cdots,z_{n-1},z_n,z_{n+1})\\&=\{p_j(z_1,\cdots,z_{n-1},z_{n+1})-p_j(z_1,\cdots,z_{n-1},z_n)\}\div(z_{n+1}-z_n).\end{aligned}$$

We are now ready to carry out Jacobi's transformation of the symmetric polynomial $A(v_1, \cdots, v_n) \div \Delta(\boldsymbol{z})$. The determinant $A(v_1, \cdots, v_n)$ has $z_j^{v_i} = p_{v_i}(z_j)$ as the element in the $i$-th row and $j$-th column. On subtracting the last column from each of the others and removing the factors

$$(z_1 - z_n)(z_2 - z_n) \cdots (z_{n-1} - z_n)$$

we obtain an $n$-th order determinant of which the element in the $i$-th row and $j$-th column is $p_{v_i-1}(z_j, z_n)$ $(j = 1, \cdots, n-1)$; the element in the $i$-th row and $n$-th column remaining $p_{v_i}(z_n)$. We now subtract the $(n-1)$-st column from each of the columns which precede it and remove the factors

$$(z_1 - z_{n-1})(z_2 - z_{n-1}) \cdots (z_{n-2} - z_{n-1})$$

obtaining a determinant of which the element in the $i$-th row and $j$-th column is $p_{v_i-2}(z_j, z_{n-1}, z_n)$ $(j = 1, \cdots, n-2)$, the elements in the last two columns remaining $p_{v_i-1}(z_{n-1}, z_n)$ and $p_{v_i}(z_n)$. Proceeding in this way we see that $\{\lambda\}(\boldsymbol{z}) \equiv A(v_1, \cdots, v_n) \div \Delta(\boldsymbol{z})$ may be expressed as a determinant of order $n$ of which the element in the $i$-th row and $j$-th column is $p_{v_i-(n-j)}(z_j, z_{j+1}, \cdots, z_n)$ $(j = 1, 2, \cdots, n)$, (it being always understood that the $p$'s with negative labels vanish). Upon multiplying this determinant by unity in the form of an $n$-th order determinant of which the element in the $i$-th row and $j$-th column is $p_{j-i}(z_1, \cdots, z_i)$ (so that the diagonal elements are all unity whilst the elements below the diagonal vanish) and using (8) (the multiplication is done row into column as in matrix multiplication) we find that

$$A(v_1, \cdots, v_n) \div \Delta(\boldsymbol{z})$$

may be expressed as an $n$-th order determinant of which the element in the $i$-th row and $j$-th column is $p_{v_i-(n-j)}(\boldsymbol{z}) = q_{v_i-(n-j)}(\boldsymbol{s})$. On setting

$$v_i = \lambda_i + n - i, \qquad (i = 1, \cdots, n)$$

we see that $\{\lambda\}(\boldsymbol{z})$ is expressible as an $n$-th order determinant whose diagonal elements are $q_{\lambda_i}(\boldsymbol{s})$ the other elements in any row being obtained by methodically increasing (decreasing) the suffix carried by $q(\boldsymbol{s})$ as we move from any column to its neighbor on the right (left). If $k$ is such that $\lambda_k > 0$ whilst $\lambda_{k+1} = \lambda_{k+2} = \cdots = \lambda_n = 0$ the last $n - k$ rows of our determinant have unity in the diagonal and zero's preceding the diagonal. Hence, and this is the

essential simplification, $\{\lambda\}(\boldsymbol{z})$ may be expressed as a determinant of order $k$ of the type described above. The coefficient of $s_1{}^n$ is obtained by setting $s_1 = 1, s_2 = \cdots = s_n = 0$ and turns out to be positive (the calculation having been performed already on p. 453).

We restate the main theorem (Frobenius-Schur) of the present section as follows: Attached to each partition $(\lambda)$ of $n$: $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$, is an irreducible representation $D(\lambda)$ of the symmetric group on $n$ letters. Its characteristic is the determinant of order $k$

$$(10) \qquad \{\lambda\}(\boldsymbol{z}) = \phi_{(\lambda)}(\boldsymbol{s}) = \begin{vmatrix} q_{\lambda_1}(\boldsymbol{s}) & \cdots & \cdot \\ \cdot & q_{\lambda_2}(\boldsymbol{s}) & \cdot \\ \cdot & \cdots & q_{\lambda_k}(\boldsymbol{s}) \end{vmatrix}$$

(where the remaining elements of any row are obtained from the diagonal element by methodically increasing (decreasing) by unity the suffix carried by $q(\boldsymbol{s})$ as we move from any column to its neighbor on the right (left)). The characteristic of the irreducible representation $D(\mu)$ which is attached to the associated partition $(\mu)$ of $n$ is

$$\phi_{(\mu)}(\boldsymbol{s}) = \begin{vmatrix} q_{\mu_1}(\boldsymbol{s}) & \cdots & \cdot \\ \cdot & q_{\mu_2}(\boldsymbol{s}) & \cdot \\ \cdot & \cdots & q_{\mu_j}(\boldsymbol{s}) \end{vmatrix} = \begin{vmatrix} \pi_{\lambda_1}(\boldsymbol{s}) & \cdots & \cdot \\ \cdot & \pi_{\lambda_2}(\boldsymbol{s}) & \cdot \\ \cdot & \cdots & \pi_{\lambda_k}(\boldsymbol{s}) \end{vmatrix}$$

so that $\phi_{(\mu)}(\boldsymbol{s}) = \phi_{(\lambda)}(s_1, -s_2, s_3, -s_4, \cdots)$. In other words, the characters of $D(\mu)$ which correspond to *even* classes are the *same* as the characters of $D(\lambda)$ whilst those which correspond to *odd* classes are the *negatives* of the characters of $D(\lambda)$; so that in constructing the character tables it is unnecessary to give the characters of $D(\mu)$ if those of $D(\lambda)$ have been already given. The common dimension of the irreducible representations $D(\lambda)$, $D(\mu)$ is

$$(11) \qquad n!\,\Delta(l) \div l_1!\,l_2!\cdots l_k!$$

where

$$l_1 = \lambda_1 + (k-1),\ l_2 = \lambda_2 + (k-2), \cdots, l_k = \lambda_k \text{ and } \Delta(l) = \prod_{p<q} (l_p - l_q).$$

**The construction of the character tables for the various symmetric groups.** From the expression (10) for $\phi_\lambda(\boldsymbol{s})$ and the relations (5) it follows at once, on applying the rule for differentiating a determinant, that $p\partial\phi_{(\lambda)}(\boldsymbol{s})/\partial s_p$ is the sum of $k$ determinants of which the $j$-th differs from $\phi_{(\lambda)}(\boldsymbol{s})$ in that the suffixes of the $q$'s in the $j$-th row are all diminished by $p$; $(p = 1, \cdots, n)$.

The suffixes of the diagonal elements of this $j$-th determinant, namely $(\lambda_1, \lambda_2, \cdots, \lambda_j - p, \cdots, \lambda_k)$ add up to $n - p$ but they will not, in general, constitute a partition of $n - p$ for $\lambda_j - p$ may well be negative and even if it is not the normal non-increasing order may well be destroyed. However, an interchange of two adjacent rows of our determinant, which amounts only to a change of its sign, changes two adjacent diagonal suffixes by interchanging them and at the same time decreasing the one which was originally on the right by unity and increasing the one which was originally on the left by unity. By doing this sufficiently often the sequence $(\lambda_1, \lambda_2, \cdots, \lambda_j - p, \cdots, \lambda_k)$ may be put in non-ascending order. If it then *ends* in a negative integer we discard the corresponding determinant, whose last row consists entirely of zeros; if it ends in one or more zeros we ignore *these* as the corresponding determinant has units in the diagonal places in the last one or more rows, all preceding elements in these rows being zero. We shall understand by $\{\lambda_1, \lambda_2, \cdots, \lambda_j - p, \cdots, \lambda_k\}$ the simple characteristic of the symmetric group on $n - p$ letters $(p = 1, 2, \cdots, n - 1)$ corresponding to the partition of $n - p$ obtained in this way *provided the number of necessary interchanges is even* and the negative of this simple characteristic of the number of interchanges is *odd*. Since $\{\cdots a, b, \cdots\} = -\{\cdots b - 1, a + 1 \cdots\}$ it is clear that $\{\cdots a, b \cdots\} = 0$ if $b = a + 1$; similarly $\{\cdots a, b, c, d \cdots\} = 0$ if $c = a + 2$ or if $d = a + 3$ and so on. With this understanding of the symbol $\{\lambda_1, \lambda_2, \cdots, \lambda_j - p, \cdots, \lambda_k\}$ we have, then,

$$p \frac{\partial \phi_{(\lambda)}(s)}{\partial s_p} = \sum_{j=1}^{k} \{\lambda_1, \lambda_2, \cdots, \lambda_j - p, \cdots, \lambda_k\} \cdots. \tag{12}$$

On writing out $\phi_\lambda(s)$ thus:

$$\phi_{(\lambda)}(s) = \sum_{(\alpha)} \frac{\chi_{(\lambda)}{}^{(\alpha)}}{\alpha_1!\,\alpha_2!\cdots\alpha_n!} \left(\frac{s_1}{1}\right)^{\alpha_1} \cdots \left(\frac{s_n}{n}\right)^{\alpha_n}$$

we have

$$p \frac{\partial \phi_{(\lambda)}(s)}{\partial s_p} = \sum_{(\alpha')} \frac{\chi_{(\lambda)}{}^{(\alpha)}}{\alpha_1!\cdots\alpha'_p!\cdots\alpha_n!} \left(\frac{s_1}{1}\right)^{\alpha_1} \cdots \left(\frac{s_p}{p}\right)^{\alpha'_p} \cdots \left(\frac{s_n}{n}\right)^{\alpha_n}$$

where $\alpha'_p = \alpha_p - 1$ and $(\alpha') = (\alpha_1, \cdots, \alpha'_p, \cdots, \alpha_n)$ is that class of the symmetric group on $n - p$ letters which contains one less cycle on $p$ letters than the class $(\alpha)$ of the symmetric group on $n$ letters. On equating coefficients of $s_1{}^{\alpha_1} \cdots s_p{}^{\alpha'_p} \cdots s_n{}^{\alpha_n}$ on both sides of the equation (12) we find

$$\chi_{(\lambda)}{}^{(\alpha)} = \sum_{j=1}^{k} \chi^{(\alpha')}_{(\lambda_1, \ldots, \lambda_j - p, \ldots, \lambda_k)}$$

a relation which we find convenient to write in the form

$$(13)\qquad \{\lambda_1, \cdots, \lambda_k\}_{(a)} = \sum_{j=1}^{k} \{\lambda_1, \lambda_2, \cdots, \lambda_j - p, \cdots, \lambda_k\}_{(a')}.$$

This basic formula enables us to write down at once those characters of the symmetric group on $n$ letters which correspond to a class containing at least one cycle on $p$ letters when the characters of that class of the symmetric group on $n-p$ letters which contains one less cycle on $p$ letters are known; $(p = 1, \cdots, n-1)$. The same formula yields directly the characters of the class containing but one cycle on $n$ letters; since $\lambda_k \geq 1$, $\lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_k \geq 1$ we have $\lambda_1 + (k-1) \leq n$ the equality holding only when $\lambda_2 = \lambda_3 = \cdots = \lambda_k = 1$ and so $\lambda_1 - n + (k-1) \leq 0$ and this implies $\{\lambda_1 - n, \lambda_2, \cdots, \lambda_k\} = 0$ unless $\lambda_2 = \lambda_3 = \cdots = \lambda_k = 1$ since then the last term, when it is rearranged in non-increasing order, namely $\lambda_1 - n + k - 1, < 0$. The other terms $\{\lambda_1, \lambda_2 - n, \lambda_3, \cdots, \lambda_k\}$ etc., are zero for *all* partitions $(\lambda)$ since $\lambda_2 - n + (k-2) < \lambda_1 - n + k - 1 < 0$ and so on. Hence the characters of the class containing but one cycle on $n$ letters are zero unless the partition $(\lambda)$ is of the type $(n-k+1, 1^{k-1})$. On subtracting $n$ from the first number $n-k+1$ of this partition of $n$ we obtain $\{1-k, 1^{k-1}\}$ and $k-1$ rearrangements are necessary to write this as $\{0^k\}$ which $= 1$. Since

$$n\,\frac{\partial \phi_{(\lambda)}(s)}{\partial s_n} = \chi_{(\lambda)}{}^{a_n=1}$$

we have

$$(14)\qquad \chi^{a_n=1}_{(n-k+1,\,1^{k-1})} = (-1)^{k-1}; \quad \text{all other } \chi_{(\lambda)}{}^{a_n=1} = 0.$$

This formula has the definite advantage, over the recurrence formula (13), that it tells us explicitly, *without referring to data concerning the symmetric group on a lesser number of letters,* the characters attached to a *particular class* of the symmetric group on $n$ letters, namely the class containing but one cycle on $n$ letters. The formula (11) of Frobenius giving the dimension of $D(\lambda)$, or, equivalently, the character attached to the unit class, has a similar advantage. We may combine our recurrence formula with the dimension formula of Frobenius to determine *directly* characters of classes containing one or more unary cycles. E. g., suppose we wish to calculate the characters of the symmetric group on $n = 20$ letters corresponding to the class containing $\alpha_1 = 12$ unary and $\alpha_8 = 1$ cycle on 8 letters. We shall illustrate by considering the representation $D(9, 6, 3, 2)$. Applying our recurrence formula with $p = 8$ we obtain

$$\{9,6,3,2\}_a = \{1,6,3,2\}_{a'} + \{9,-2,3,2\}_{a'} + \{9,6,-5,2\}_{a'} + \{9,6,3,-6\}_{a'};$$

of the four terms on the right the first, third and fourth vanish; the first because $3 = 1 + 2$; the third because $-5 + 1 < 0$ and the fourth because $-6 < 0$. There remains $\{9,-2,3,2\}_{a'} = -\{9,2,-1,2\}_{a'} = \{9,2,1\}_{a'}$ and since $\alpha'$ is the unit class the dimension formula of Frobenius yields, since

$$(l_1, l_2, l_3) = (11, 3, 1), \qquad \frac{12\,!}{11\,!\,3\,!\,1\,!}(8)(10)(2) = 320.$$

Similar, although not quite such convenient, formulae may be found for the characters of a class containing only cycles of the same length. E. g., let $n = 2m$ and consider the class containing $m$ binary cycles. The characters of this class are found by setting $s_2 = 1$, $s_1 = s_3 = \cdots = s_n = 0$ in the expressions for the simple characteristics; it being clear that then $q_j = 0$ if $j$ is odd whilst $q_{2p} = 1/2^p.\,p\,!$. Thus, for $n = 12$, the character of the class $\alpha_2 = 6$ of $D(5,4,2,1)$ is

$$2^6.\,6\,!\begin{vmatrix} 0 & (2^3.\,3\,!)^{-1} & 0 & (2^4.\,4\,!)^{-1} \\ 0 & (2^2.\,2\,!)^{-1} & 0 & (2^3.\,3\,!)^{-1} \\ 1 & 0 & 2^{-1} & 0 \\ 0 & 0 & 1 & 0 \end{vmatrix} = 6\,!\,\{(4\,!\,2\,!)^{-1} - (3\,!)^{-2}\} = -5.$$

Similarly the character of the class $\alpha_3 = 4$ of $D(6, 3^2)$ is

$$3^4.\,4\,!\begin{vmatrix} (3^2.\,2\,!)^{-1} & 0 & 0 \\ 0 & 3^{-1} & 0 \\ 0 & 0 & 3^{-1} \end{vmatrix} = 12$$

whilst the character of the class $\alpha_4 = 3$ of $D(7, 2^2, 1)$ is

$$4^3.\,3\,!\begin{vmatrix} 0 & (4^2.\,2\,!)^{-1} & 0 & 0 \\ 0 & 0 & 0 & 4^{-1} \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{vmatrix} = -3.$$

Two more examples will suffice; suppose for $n = 15$ we wish the character of the class $\alpha_5 = 3$ for the representation $D(5,4,3,2,1)$. On setting $s_5 = 1$ $s_1 = s_2 = \cdots = s_{15} = 0$ all the $q_j$ vanish save those for which $j$ is a multiple of 5 and $q_{5p}$ takes the value $(5^p.\,p\,!)^{-1}$. Then the desired character $=$

$$5^3 . 3\,! \begin{vmatrix} 5^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{vmatrix} = -150.$$

If we wish, for a final example, to obtain for $n = 12$ the character of the class $\alpha_1 = 1$, $\alpha_2 = 2$, $\alpha_3 = 3$ for the representation $D(7, 1^5)$ we may proceed as follows. On applying our recurrence formula twice, first with $p = 1$ and then with $p = 2$, we find

$$\begin{aligned} \{7, 1^5\}_{\alpha} &= \{6, 1^5\}_{\alpha'} + \{7, 1^4\}_{\alpha'} \\ &= \{4, 1^5\}_{\alpha''} - \{6, 1^3\}_{\alpha''} + \{5, 1^4\}_{\alpha''} - \{7, 1^2\}_{\alpha''} \end{aligned}$$

where $\alpha''$ is the class, of the symmetric group on 9 letters, consisting of permutations each of which has three ternary cycles. Since this class is positive $\{4, 1^5\}_{\alpha''} = \{6, 1^3\}_{\alpha''}$ and we have merely to calculate $\{5, 1^4\}_{\alpha''}$ and $\{7, 1^2\}_{\alpha''}$. We find

$$\{5, 1^4\}_{\alpha''} = 3^3 . 3\,! \begin{vmatrix} 0 & (3^2 . 2\,!)^{-1} & 0 & 0 & (3^3 . 3\,!)^{-1} \\ 1 & 0 & 0 & 3^{-1} & 0 \\ 0 & 1 & 0 & 0 & 3^{-1} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{vmatrix} = -2$$

$$\{7, 1^2\}_{\alpha''} = 3^3 . 3\,! \begin{vmatrix} 0 & 0 & (3^3 . 3\,!)^{-1} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 1$$

so that the desired character is $-3$. The most trivial example of this method furnishes directly the characters of the class $\alpha_1 = 1$, $\alpha_{n-1} = 1$ of the symmetric group on $n$ letters. All characters are zero save that of the identical representation and those associated with the partitions

$$\lambda_1 = n - k, \qquad \lambda_2 = 2, \qquad \lambda_3 = \cdots = \lambda_k = 1$$

and these have the value $(-1)^{k-1}$.

The character tables for the various symmetric groups from $n = 2$ to $n = 10$, inclusive, are given in the paper numbered **(15)** in the list of references. The character table for $n = 11$ is given in the paper numbered **(16)**; (anyone

using this table should note that the characters of $(5, 3, 1^3)$ are given with the wrong signs for the odd classes). And the character tables for $n = 12$ and $n = 13$ are given in the paper numbered **(17)**.

We may derive by the method just described explicit formulae for the characters of those classes of the symmetric group on $n$ letters which consist of $\alpha_1 = n - p$ unary cycles and $\alpha_p = 1$ cycle on $p$ letters; $(p = 2, 3, 4, \cdots)$. These formulae were given by Frobenius **(4)** for $p = 2$ (the transposition class) and **(5)** for $p = 3, 4$. Since they are of importance in the physical applications we give their derivation here. We first remark that a partition $(\lambda)$ of $n$ may be conveniently specified as follows: draw the principal diagonal of the diagram of the partition (i. e. the diagonal starting at the upper left-hand corner) and suppose it strikes $s$ columns. Denote by $b_1 > b_2 > \cdots > b_s \geq 0$ the number of dots to the right of the diagonal in the rows $1, 2, \cdots, s$, respectively, and by $a_1 > a_2 > \cdots > a_s \geq 0$ the number of dots below the diagonal in the columns $1, \cdots, s$ respectively. Then the partition is described by $\boldsymbol{b} = (b_1, \cdots, b_s)$ and $\boldsymbol{a} = (a_1, \cdots, a_s)$ it being clear that the partition is self-associated when and only when $\boldsymbol{a} = \boldsymbol{b}$. The number of dots in the first row and column together $= b_1 + a_1 + 1$; when these are deleted the number of dots in the new first row and column $= b_2 + a_2 + 1$. Proceeding in this way we have $n = \sum_{j=1}^{s} (b_j + a_j + 1)$. It is clear from the definition that $b_j = \lambda_j - j$ $(j = 1, \cdots, s)$, whilst the differences $p - \lambda_{p+1}$ $(p = s, \cdots, k - 1)$, satisfy the inequalities

$$0 \leq s - \lambda_{s+1} < s + 1 - \lambda_{s+2} < \cdots < k - 1 - \lambda_k < k - 1.$$

Hence they are the complementary set to the set $a_1 > a_2 > \cdots > a_s$ in the set $0, 1, \cdots, k - 1$. In fact $\lambda_k > 0$ shows that $a_1 = k - 1$ is not in the set; if $\lambda_k > 1$, $a_2 = k - 2$ is not in the set and so on. The following will serve as illustrations of the definitions of $\boldsymbol{b}$ and $\boldsymbol{a}$:

$$(\lambda) = (3, 2^2, 1^2);\ s = 2;\ \boldsymbol{b} = (2, 0);\quad \boldsymbol{a} = (4, 1)$$
$$(\lambda) = (4, 2, 1^2);\ s = 2;\ \boldsymbol{b} = (3, 0);\quad \boldsymbol{a} = (3, 0)$$
$$(\lambda) = (4^3, 1);\quad s = 3;\ \boldsymbol{b} = (3, 2, 1);\ \boldsymbol{a} = (3, 1, 0).$$

We denote, for convenience, by $\chi_{(\lambda)}(p)$ the characters of the class $\alpha_1 = n - p$, $\alpha_p = 1$ so that, for instance, $\chi_{(\lambda)}(2)$ are the characters of the transposition class whilst $\chi_{(\lambda)}(1)$ are the characters of the unit class (i. e. the dimensions

of the various irreducible representations). Our object is to obtain for $\chi_{(\lambda)}(p)$ $(p=2,3,4,\cdots)$, an explicit formula analogous to (11) which furnishes $\chi_{(\lambda)}(1)$. The recurrence formula (13) tells us that $\chi_{(\lambda)}(p)$ is the sum of the dimensions of the irreducible representations

$$D(\lambda_1,\lambda_2,\cdots,\lambda_j-p,\cdots,\lambda_k) \qquad (j=1,\cdots,k),$$

of the symmetric group on $n-p$ letters (where we follow the previously agreed on convention for the restoration of the normal non-increasing order of the $(\lambda_1,\lambda_2,\cdots)$ when this has been destroyed by the subtraction of $p$). Writing, as before,

$$l_1=\lambda_1+(k-1), \qquad l_2=\lambda_2+(k-2),\cdots,l_k=\lambda_k$$

the dimension of $D(\lambda_1-p,\lambda_2,\cdots,\lambda_k)$ is

$$(n-p)!\,(l_1-p-l_2)\cdots(l_1-p-l_k)(l_2-l_3)\cdots(l_{k-1}-l_k)$$
$$\div\,(l_1-p)!\,l_2!\cdots l_k!$$

and there are similar expressions of the dimensions of the other irreducible representations. On dividing through by

$$\chi_{(\lambda)}(1)=n!\,(l_1-l_2)\cdots(l_{k-1}-l_k)\div l_1!\cdots l_k!$$

the quotient $\chi_{(\lambda)}(p)\div\chi_{(\lambda)}(1)$ appears as a sum of $k$ terms of which the first is

$$l_1(l_1-1)\cdots(l_1-p+1)(l_1-p-l_2)\cdots(l_1-p-l_k)$$
$$\div\,n(n-1)\cdots(n-p+1)(l_1-l_2)\cdots(l_1-l_k).$$

If we write $f(x)\equiv(x-l_1)\cdots(x-l_k)$ this may be written as the quotient of $l_1(l_1-1)\cdots(l_1-p+1)f(l_1-p)$ by $-pn(n-1)\cdots(n-p+1)f'(l_1)$ where $f'$ indicates the derivative of $f$; hence

$$\frac{\chi_{(\lambda)}(p)}{\chi_{(\lambda)}(1)}=\frac{-1}{pn(n-1)\cdots(n-p+1)}\sum_{j=1}^{k}\frac{l_j(l_j-1)\cdots(l_j-p+1)f(l_j-p)}{f'(l_j)}.$$

Now the analysis of the function $x(x-1)\cdots(x-p+1)f(x-p)\div f(x)$ into simple fractions yields a polynomial in $x$ plus terms $A_j\div(x-l_j)$ where $A_j=l_j(l_j-1)\cdots(l_j-p+1)f(l_j-p)\div f'(l_j)$ so that

$$\chi_{(\lambda)}(p)\div\chi_{(\lambda)}(1)=-\left(\sum_{j=1}^{k}A_j\right)\div pn(n-1)\cdots(n-p+1).$$

On writing $(x-l_j)^{-1}=(1/x)+(l_j/x^2)+\cdots$ it is clear that $\sum_{j=1}^{k}A_j$ is the coefficient of $(1/x)$ in the development of

$$x(x-1)\cdots(x-p+1)f(x-p)\div f(x)$$

as a series of *descending* powers of $x$. The zeros of $f(x)$ are the $k$ numbers $(l_1,\cdots,l_k)$ so that, if $y=x-k$, the zeros of $f(y+k)$ are the $k$ numbers $l_1-k,\cdots,l_k-k$ i. e. the $k$ numbers $\lambda_j-j$ $(j=1,\cdots,k)$. Of these the first $s$ are the numbers $(b_1,\cdots,b_s)$ whilst the remaining $k-s$ are the negatives of $\alpha_j+1$ where the two sets $a=(a_1,\cdots,a_s)$ and $\alpha=(\alpha_{s+1},\cdots,\alpha_k)$ together form the set $0,1,\cdots,(k-1)$. Hence

$$f(y+k)=\prod_{j=1}^{s}(y-b_j)\cdot\prod_{s+1}^{k}(y+\alpha_h+1)$$
$$=\prod_{j=1}^{s}\{(y-b_j)/(y+a_j+1)\}.\,(y+1)\cdots(y+k).$$

It will be convenient to denote the function

$$(y-b_1)\cdots(y-b_s)/(y+a_1+1)\cdots(y+a_s+1)$$

by $F(y)$ and then $f(x)=f(y+k)=F(y)(y+1)\cdots(y+k)$. The desired sum $\sum_{j=1}^{k}A_j$, being the coefficient of $1/x$ in the development of

$$x(x-1)\cdots(x-p+1)f(x-p)\div f(x)$$

in a series of descending powers of $x$, is, equivalently, the coefficient of $1/y$ in the development of this same function in a descending series of powers of $y$. But

$$\begin{aligned}&x(x-1)\cdots(x-p+1)f(x-p)\div f(x)\\&\quad=(y+k)\cdots(y+k+1-p)\cdot(y+k-p)\cdots(y+1-p)F(y-p)\\&\qquad\div(y+1)\cdots(y+k)F(y)\\&\quad=y(y-1)\cdots(y-p+1)F(y-p)\div F(y)\end{aligned}$$

and we have merely to seek the coefficient of $(1/y)$ in the development of this function. An application of Taylor's expansion yields

$$\begin{aligned}F(y-p)\div F(y)=1-pF'(y)/F(y)+p^2F''(y)/2!\,F(y)\\-p^3F'''(y)/3!\,F(y)+\cdots;\end{aligned}$$

on taking the logarithmic derivative of

$$F(y)=\prod_{j=1}^{s}\{(y-b_j)/(y+a_j+1)\}$$

we find

$$F'(y)/F(y) = \sum_{j=1}^{s} \{[1/(y-b_j)] - [1/(y+a_j+1)]\}$$
$$= (n/y^2) + (c_3/y^3) + (c_4/y^4) + \cdots$$

where

$$c_3 = \sum_{j=1}^{s} \{b_j^2 - (a_j+1)^2\};\ c_4 = \sum_{j=1}^{s} \{b_j^3 + (a_j+1)^3\};\ c_5 = \sum_{j=1}^{s} \{b_j^4 - (a_j+1)^4\} \cdots$$

(we have availed ourselves of the relation $\sum_{j=1}^{s} \{b_j + (a_j+1)\} = n$). On successive differentiation of this relation we find

$$F''(y)/F(y) = \{F'(y)/F(y)\}^2 + \{F'(y)/F(y)\}'$$
$$= (-2n/y^3) + \{(n^2 - 3c_3)/y^4\} + \{(2nc_3 - 4c_4)/y^5\} + \cdots$$
$$F'''(y)/F(y) = \{F''(y)/F(y)\}\{F'(y)/F(y)\} + \{F''(y)/F(y)\}'$$
$$= (6n/y^4) + \{(12c_3 - 6n^2)/y^5\} + \cdots$$
$$F''''(y)/F(y) = \{F'''(y)/F(y)\}\{F'(y)/F(y)\} + \{F'''(y)/F(y)\}'$$
$$= (-24n/y^5) + \cdots.$$

Hence

$$F(y-p) \div -pF(y) = -(1/p) + (n/y^2) + (pn + c_3)/y^3$$
$$+ (2c_4 + 3pc_3 + 2np^2 - pn^2)/y^4$$
$$+ \{c_5 + 2pc_4 + p(2p-n)c_3 + np^2(p-n)\}/y^5 \cdots.$$

This has to be multiplied by $y(y-1)\cdots(y-p+1)$ and the coefficient of $y^{-1}$ in the product then determined; equivalently we may multiply by $(y-1)\cdots(y-p+1)$ and determine the coefficient of $y^{-2}$. This coefficient yields, when divided by $n(n-1)\cdots(n-p+1)$ the desired quantity $\chi_{(\lambda)}(p) \div \chi_{(\lambda)}(1)$. We carry out the calculation for $p = 2, 3, 4$.

$$p = 2; \qquad \chi_{(\lambda)}(2) \div \chi_{(\lambda)}(1) = (n + c_3) \div n(n-1).$$

Since

$$c_3 = \sum_{j=1}^{s} \{b_j^2 - (a_j+1)^2\}, \qquad n = \sum_{j=1}^{s} \{b_j + (a_j+1)\}$$
$$c_3 + n = \sum_{j=1}^{s} \{b_j(b_j+1) - a_j(a_j+1)\}$$

so that

$$(15) \quad \chi_{(\lambda)}(2) \div \chi_{(\lambda)}(1) = [\sum_{j=1}^{s} \{b_j(b_j+1) - a_j(a_j+1)\}] \div n(n-1)$$

$p = 3$; here we must multiply by $(y-1)(y-2)$ and the coefficient of $y^{-2}$ is

$$(2c_4 + 3c_3 - 3n^2 + 4n)/2 = \tfrac{1}{2}[\sum_{j=1}^{s}\{(2b_j^3 + 3b_j^2 + b_j) + 2(a_j + 1)^3 - 3(a_j + 1)^2 + (a_j + 1)\} - 3n(n-1)].$$

Hence

$$(16) \quad \chi_{(\lambda)}(3) \div \chi_{(\lambda)}(1) = [\sum_{j=1}^{s}\{b_j(b_j + 1)(2b_j + 1) + a_j(a_j + 1)(2a_j + 1)\} - 3n(n-1)] \div 2n(n-1)(n-2)$$

$p = 4$; here we must multiply by

$$(y-1)(y-2)(y-3) = y^3 - 6y^2 + 11y - 6$$

and the coefficient of $y^{-2}$ is $c_5 + 2c_4 + c_3 - 2(2n-3)(c_3 + n)$

$$= \sum_{j=1}^{s}[\{b_j^2(b_j + 1)^2 - a_j^2(a_j + 1)^2\} - 2(2n-3)\{b_j(b_j + 1) - a_j(a_j + 1)\}]$$

so that

$$(17) \quad \chi_{(\lambda)}(4) \div \chi_{(\lambda)}(1) = [\sum_{j=1}^{s}\{b_j^2(b_j + 1)^2 - a_j^2(a_j + 1)^2\} - 2(2n-3)\{b_j(b_j+1) - a_j(a_j+1)\}] \div n(n-1)(n-2)(n-3).$$

For higher values of $p$ it is more serviceable to use the recurrence formula (13) as the expressions deduced by the manner described above become too complicated. The formula (15) of Frobenius may be readily transformed into an equivalent formula due to Hund (see reference (**23**)). We have $b_j = \lambda_j - j$, $(j = 1, \cdots, s)$, $\alpha_j + 1 = j - \lambda_j$, $(j = s + 1, \cdots, k)$ where $\boldsymbol{a} = (a_1, \cdots, a_s)$ and $\boldsymbol{\alpha} = (\alpha_{s+1}, \cdots, \alpha_k)$ together form the set $(0, \cdots, k-1)$; so that

$$\sum_{1}^{s} a_j(a_j + 1) = \sum_{0}^{k-1} p(p+1) - \sum_{s+1}^{k} \alpha_j(\alpha_j + 1)$$
$$= \sum_{1}^{k} p(p-1) - \sum_{s+1}^{k}(\lambda_j - j)(\lambda_j - j + 1).$$

Hence

$$\sum_{1}^{s}\{b_j(b_j + 1) - a_j(a_j + 1)\} = \sum_{1}^{k}(\lambda_j - j)(\lambda_j - j + 1) - \sum_{1}^{k} p(p-1)$$
$$= \sum_{1}^{k} \lambda_j(\lambda_j - 2j + 1)$$

so that

$$\chi_{(\lambda)}(2) \div \chi_{\lambda}(1) = \sum_{1}^{k} \lambda_j(\lambda_j - 2j + 1) \div n(n-1)$$

which is Hund's formula.

**The analysis of the reducible representations $\Delta(\lambda_1, \cdots, \lambda_k)$ into irreducible components.** The characteristic of $\Delta(\lambda_1, \cdots, \lambda_k)$ is $q_{\lambda_1}(\boldsymbol{s}) \cdots q_{\lambda_k}(\boldsymbol{s})$ whilst that of the irreducible representation $D(\lambda_1, \cdots, \lambda_k)$ is

$$\{\lambda\}(s) = \begin{vmatrix} q_{\lambda_1}(s) & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & q_{\lambda_k}(s) \end{vmatrix};$$

the problem confronting us is that of writing $q_{\lambda_1}(s)\cdots q_{\lambda_k}(s)$ as a linear combination, with positive or zero integral coefficients, of the various simple characteristics $\{\lambda\}(s)$. When $k=2$ the solution is trivially evident:

$$q_{\lambda_1}(s)\,q_{\lambda_2}(s) = \begin{vmatrix} q_{\lambda_1}(s) & q_{\lambda_1+1}(s) \\ q_{\lambda_2-1}(s) & q_{\lambda_2}(s) \end{vmatrix} + \begin{vmatrix} q_{\lambda_1+1}(s) & q_{\lambda_1+2}(s) \\ q_{\lambda_2-2}(s) & q_{\lambda_2-1}(s) \end{vmatrix} + \cdots$$

so that

$$\begin{aligned} (18)\quad \Delta(\lambda_1,\lambda_2) = D(\lambda_1,\lambda_2) &+ D(\lambda_1+1,\lambda_2-1) \\ &+ D(\lambda_1+2,\lambda_2-2) + \cdots + D(n). \end{aligned}$$

For $k>2$ the problem may be solved as follows (we illustrate by considering the case $k=3$). Let $x_j$ be an operator whose effect is to replace $q_{\lambda_j}(s)$ by $q_{\lambda_j+1}(s)$ : $x_j q_{\lambda_j}(s) = q_{\lambda_j+1}(s)$ $(j=1,2,3)$. Then the determinant

$$\{\lambda\}(s) = \begin{vmatrix} q_{\lambda_1}(s) & q_{\lambda_1+1}(s) & q_{\lambda_1+2}(s) \\ q_{\lambda_2-1}(s) & q_{\lambda_2}(s) & q_{\lambda_2+1}(s) \\ q_{\lambda_3-2}(s) & q_{\lambda_3-1}(s) & q_{\lambda_3}(s) \end{vmatrix}$$

may be expressed as the result of operating with

$$\begin{vmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{vmatrix}$$

upon the simple product $q_{\lambda_1}(s)q_{\lambda_2-1}(s)q_{\lambda_3-2}(s)$ and since the operators $x_j$ operate on different symbols, $x_j$ operating on $q_{\lambda_j}$, they are commutative so that we may apply the ordinary rules of commutative algebra. Thus

$$\{\lambda\}(s) = (x_2-x_1)(x_3-x_1)(x_3-x_2)q_{\lambda_1}(s)q_{\lambda_2-1}(s)q_{\lambda_3-2}(s)$$

and so

$$q_{\lambda_1}(s)q_{\lambda_2-1}(s)q_{\lambda_3-2}(s) = (x_2-x_1)^{-1}(x_3-x_1)^{-1}(x_3-x_2)^{-1}\{\lambda\}(s).$$

We write now $\xi_j = x_j^{-1}$ $(j=1,2,3)$, so that $\xi_j$ operates on $\lambda_j$ so as to *decrease* it by unity: $\xi_j q_{\lambda_j} = q_{\lambda_j-1}$. Then

$$(x_2-x_1)^{-1} = \xi_2(1-\xi_2 x_1)^{-1};\qquad (x_3-x_1)^{-1} = \xi_3(1-\xi_3 x_1)^{-1};$$
$$(x_3-x_2)^{-1} = \xi_3(1-\xi_3 x_2)^{-1}$$

and so

$$q_{\lambda_1}(s)\,q_{\lambda_2}(s)\,q_{\lambda_3}(s) = x_2x_3^2q_{\lambda_1}(s)\,q_{\lambda_2-1}(s)\,q_{\lambda_3-2}(s)$$
$$= (1-\xi_2x_1)^{-1}(1-\xi_3x_1)^{-1}(1-\xi_3x_2)^{-1}\{\lambda\}(s).$$

The product

$$(1-\xi_3x_1)^{-1}(1-\xi_3x_2)^{-1} = 1 + p_1(x_1,x_2)\xi_3 + p_2(x_1,x_2)\xi_3^2 + \cdots$$

and the series may be stopped at $p_{\lambda_3}(x_1,x_2)\xi_3^{\lambda_3}$ since each $\{\lambda\}$ with a negative number at the end vanishes. Then this must be multiplied by

$$(1-\xi_2x_1)^{-1} = 1 + p_1(x_1)\xi_2 + \cdots$$

and this series may be stopped, for each $\{\lambda'\}$, at $p_{\lambda'_2+1}(x_1)\xi_2^{\lambda'_2+1}$ since each $\{\lambda\}$ whose next to last member $< -1$ vanishes. It is clear then that $\Delta(\lambda)$ contains $D(\lambda)$ once and contains no $D(\lambda')$ for which $(\lambda) > (\lambda')$; it being understood that $(\lambda) > (\lambda')$ when the *first* non-vanishing member of the set $\lambda_j - \lambda'_j$ is positive $(j = 1, 2, \cdots)$. The argument is evidently perfectly general; thus for $k = 4$ we first operate with $1 + p_1(x_1,x_2,x_3)\xi_4 + p_2(x_1,x_2,x_3)\xi_4^2 + \cdots$ on $\{\lambda\}$; then follow this by $1 + p_1(x_1,x_2)\xi_3 + p_2(x_1,x_2)\xi_3^2 + \cdots$ and finally by $1 + p_1(x_1)\xi_2 + p_2(x_1)\xi_2^2 + \cdots$.

The following example will sufficiently illustrate the method: consider $\Delta(4, 2^2)$. Applying $1 + p_1(x_1,x_2)\xi_3 + p_2(x_1,x_2)\xi_3^2$ to $\{4, 2, 2\}$ we obtain

$$\{4, 2^2\} + \{5, 2, 1\} + \{4, 3, 1\} + \{6, 2\} + \{5, 3\} + \{4, 4\}.$$

Applying $1 + p_1(x_1)\xi_2 + \cdots$ to each of these we obtain in turn

$$\{4,2,2\} + \{6,0,2\} + \{7,-1,2\};\ \{5,2,1\} + \{6,1^2\} + \{8,-1,1\};$$
$$\{4,3,1\} + \{5,2,1\} + \{6,1^2\} + \{8,-1,1\};\ \{6,2\} + \{7,1\} + \{8\};$$
$$\{5,3\} + \{6,2\} + \{7,1\} + \{8\};\ \{4,4\} + \{5,3\} + \{6,2\} + \{7,1\} + \{8\}$$

and adding up we find

$$\Delta(4,2^2) = D(4,2^2) + D(4,3,1) + D(4^2) + 2D(5,2,1,) + 2D(5,3)$$
$$+ D(6,1^2) + 3D(6,2) + 2D(7,1) + D(8).$$

When there are but three elements $(\lambda_1, \lambda_2, \lambda_3)$ in the partition the theory just given leads to the following convenient formula. Denoting by $\overline{12}$ the operation $1 + x_1\xi_2 + x_1^2\xi_2^2 + \cdots$ we have to apply to $(\lambda_1,\lambda_2,\lambda_3)$ the operator

$$\overline{12} + (x_2 + 2x_1\cdot\overline{12})\xi_3 + (x_2^2 + 2x_1x_2 + 3x_1^2\cdot\overline{12})\xi_3^2$$
$$+ (x_2^3 + 2x_1x_2^2 + 3x_1^2x_2 + x_1^3\cdot\overline{12})\xi_3^3 + \cdots.$$

For example let us consider the analysis of $\Delta(4^2, 2)$; the application of $\overline{12}$ gives

$$\{4^2, 2\} + \{5, 3, 2\} + \{6, 2^2\} + \{7, 1, 2\} + \{8, 0, 2\} + \{9, -1, 2\}$$
$$= \{4^2, 2\} + \{5, 3, 2\} + \{6, 2^2\} - \{8, 1^2\} - \{9, 1\};$$

$x_2\xi_3$ yields $\{4, 5, 1\} = 0$; $2x_1 \cdot \overline{12} \cdot \xi_3$ yields

$$2[\{5, 4, 1\} + \{6, 3, 1\} + \{7, 2, 1\} + \{8, 1^2\} - \{10\}]$$

$(x_2{}^2 + 2x_1x_2)\xi_3{}^2$ yields

$$\{4, 6\} + 2\{5^2\} = \{5^2\}$$

whilst, finally, $3x_1{}^2 \cdot \overline{12} \cdot \xi^2$ yields

$$3[\{6, 4\} + \{7, 3\} + \{8, 2\} + \{9, 1\} + \{10\}].$$

Combining these results we obtain

$$\begin{aligned}\Delta(4^2, 2) = D(10) &+ 2D(9, 1) + 3D(8, 2) + D(8, 1^2) \\ &+ 3D(7, 3) + 2D(7, 2, 1) + 3D(6, 4) + 2D(6, 3, 1) \\ &+ D(6, 2^2) + D(5^2) + 2D(5, 4, 1) + D(5, 3, 2) + D(4^2, 2).\end{aligned}$$

Whilst the formula just given may be regarded as a complete *theoretical* solution of the problem of analysing the reducible representation $\Delta(\lambda)$ into its irreducible components it becomes very tedious when $k$, the number of elements in the partition $(\lambda)$, $\geq 4$. Fortunately the necessary information, up to $n = 11$, is available in tables prepared by Kostka.[1] This writer was interested in the general question of symmetric functions and in the course of his investigation took up the question of expressing a product $\sigma_{\lambda_1}(z) \cdots \sigma_{\lambda_k}(z)$ of the elementary symmetric functions of $n$ variables $(z_1, \cdots, z_n)$ as a linear combination of determinants

$$\begin{vmatrix} \sigma_{n_1}(z) & \cdot & \cdots & \sigma_{n_1+j-1}(z) \\ \sigma_{n_2-1}(z) & \sigma_{n_2}(z) & \cdots & \sigma_{n_2+j-2}(z) \\ \cdot & \cdot & \cdots & \sigma_{n_j}(z) \end{vmatrix}; \qquad n_1 + n_2 + \cdots + n_j = n.$$

Since $\sigma_m(z) = \pi_m(s) = q_m(s_1, -s_2, s_3, -s_4, \cdots)$, $m \leq n$, it follows that such an analysis of the product $\sigma_{\lambda_1}(z) \cdots \sigma_{\lambda_k}(z)$ furnishes the analysis of the product $q_{\lambda_1}(s) \cdots q_{\lambda_k}(s)$ as a linear combination of the simple characteristics $\{n_1, \cdots, n_j\}$; or, equivalently, of the reducible representations $\Delta(\lambda_1, \cdots, \lambda_k)$ as a linear combination of the irreducible representations

[1] The fact that these tables, up to $n = 8$, were published in 1882 long before the representation theory of the symmetric group, and its applications, were dreamed of, recalls to mind the verse in Ecclesiastes: "Nothing under the sun is new, neither is any man able to say: Behold this is new: for it hath already gone before in the ages that were before us."

$D(n_1, \cdots, n_j)$. The paper numbered **(18)** in the list of references gives the tables for $2 \leq n \leq 8$; that numbered **(19)** the table for $n = 9$ and that numbered **(20)**, which is inaccessible to us, the tables for $10 \leq n \leq 11$. In using these tables note that the $\Delta(\lambda)$ appear at the bottom (the symbol $K$ being used instead of $\Delta$ and the partition being indicated by a suffix, a non-decreasing rather than a non-increasing order being adopted: thus $\Delta(3^2, 2)$ appears as $K_{23^2}$); the $D(\lambda)$ appear on the right (the symbol $C$ being used instead of $D$, and the partition being indicated by a suffix, the normal non-increasing order being used: thus $D(4, 2^2)$ appears as $C_{42^2}$). The tables are square but the part above the main diagonal serves another purpose and is not used in the problem that concerns us. The following method of deriving Kostka's tables (or, more particularly, that part of them which is effective for our problem) was suggested by Littlewood and Richardson **(13)**, on the assumption that the character table of the symmetric group in question is at hand. The product $q_{\lambda_1}(\boldsymbol{s}) \cdots q_{\lambda_k}(\boldsymbol{s})$, which is the characteristic of $\Delta(\lambda_1, \cdots, \lambda_k)$ is a linear combination, $c_{(\alpha)}{}^{(\lambda)} s^{(\alpha)}$, say, of the quantities $s^{(\alpha)} = s_1{}^{\alpha_1} s_2{}^{\alpha_2} \cdots s_n{}^{\alpha_n}$ and hence, by (3), a linear combination, $\sum_{(\beta)} c_{(\alpha)}{}^{(\lambda)} \chi_{(\beta)}{}^{(\alpha)} \phi_{(\beta)}(\boldsymbol{s})$ of the simple characteristics of the symmetric group on $n$ letters. Hence the coefficient of $D(\beta)$ in the analysis of $\Delta(\lambda)$ is $c_{(\alpha)}{}^{(\lambda)} \chi_{(\beta)}{}^{(\alpha)}$; in other words it is obtained by taking the indicated linear combination of those columns of the character table which correspond to classes $(j)$ for which $s^j$ appears in the product $q_{\lambda_1}(\boldsymbol{s}) \cdots q_{\lambda_k}(\boldsymbol{s})$. This method is particularly suited to those partitions $(\lambda)$ for which many of the $\lambda_j$ are unity. For instance if they are all unity $q_{\lambda_1}(\boldsymbol{s}) \cdots q_{\lambda_k}(\boldsymbol{s}) = \{q_1(\boldsymbol{s})\}^n = s_1{}^n$ and so the coefficients in the analysis of $\Delta(1^n)$ are merely the characters of the unit class; in other words the coefficient of $D(\lambda)$ in the analysis of $\Delta(1^n)$ is the dimension of $D(\lambda)$; a fact of which we have independent knowledge since $\Delta(1^n)$ is the regular representation, of dimension $n!$, of the symmetric group. For $\Delta(2, 1^{n-2})$ the product $q_{\lambda_1}(\boldsymbol{s}) \cdots q_{\lambda_k}(\boldsymbol{s})$ is $(s_1{}^n + s_1{}^{n-2} s_2) \div 2$ and hence the coefficient of $D(\lambda)$ in the analysis of $\Delta(2, 1^{n-2})$ is the mean of the characters $\chi_{(\lambda)}{}^{1^n}$, $\chi_{(\lambda)}{}^{(1^{n-2}, 2)}$ of the unit class and the transposition class, respectively. But it is clear that whilst the analysis of $\Delta(p, 1^{n-p})$ is relatively simple by this method a partition with three or more parts $\geq 2$ leads to somewhat complicated calculations. Thus to make the analysis of $\Delta(3, 2^3)$ we would have to evaluate

$$q_3 q_2{}^3 = (s_1{}^3 + 3s_1 s_2 + 2s_3)(s_1{}^2 + s_2) \div 3!(2!)^3$$

and then form the indicated combination of the many columns of the character table (of the symmetric group on 9 letters) involved. We give below the

analysis of all $\Delta(\lambda)$ for $n \leq 9$ and in making the calculations found the following method the most convenient. It rests on a knowledge of the analysis of the product $\{\lambda\}\{\mu\}$ of two simple characteristics which is given (for $\Sigma\lambda + \Sigma\mu \leq 9$) in the following section. Suppose we wish to analyse the reducible representation $\Delta(3,2,1)$ of the symmetric group on six letters. We have $q_2 q_1 = \{3\} + \{2,1\}$ so that, since $q_3 = \{3\}$, $q_3 q_2 q_1 = \{3\}\{3\} + \{3\}\{2,1\}$. From the values given at the end of the next section we read off

$$\begin{aligned} \{3\}\{3\} &= \{6\} + \{5,1\} + \{4,2\} + \{3^2\}. \\ \{3\}\{2,1\} &= \qquad\quad \{5,1\} + \{4,2\} + \{4,1^2\} + \{3,2,1\} \end{aligned}$$

so that

$$q_3 q_2 q_1 = \{6\} + 2\{5,1\} + 2\{4,2\} + \{4,1^2\} + \{3^2\} + \{3,2,1\}$$

or, equivalently,

$$\Delta(3,2,1) = D(6) + 2D(5,1) + 2D(4,2) + D(4,1^2) + D(3^2) + D(3,2,1).$$

In the following tables the irreducible representations are written across the top, the $D$ being omitted in the interest of space, and the reducible representations are written down the left. For convenience of printing, Table 8, $n = 9$, is turned around so that the bottom of the page is the left-hand side of the table and the left-hand side of the page the top of the table. As examples of how the tables are read we cite the following:

$$\begin{aligned} &n = 2;\ \Delta(1^2) = D(2) + D(1^2) \\ &n = 3;\ \Delta(2,1) = D(3) + D(2,1) \\ &n = 4;\ \Delta(2,1) = D(4) + 2D(3,1) + D(2^2) + D(2,1) \\ &n = 5;\ \Delta(2,1) = D(5) + 2D(4,1) + 2D(3,2) + D(3,1^2) + D(2^2,1) \\ &n = 6;\ \Delta(2^3) = D(6) + 2D(5,1) + 3D(4,2) \\ &\qquad\qquad + D(4,1^2) + D(3^2) + 2D(3,2,1) + D(2^3) \end{aligned}$$

The numbers to the right of the main diagonal are all zero and are not written in. It may be observed that there is considerable duplication in the tables, the coefficients of the analysis of $\Delta(\lambda)$ being independent of $n$ for the earlier partitions. Thus the coefficients in the analysis of the first twelve partitions of 9, from $(9)$ to $(5,1^4)$ inclusive, are the same as those in the analyses of the first twelve partitions of 8, from $(8)$ to $(4,1^4)$ inclusive. The table for $n = 10$ coincides with that for $n = 9$ for the first 19 partitions (from $(10)$ to $(5,1^5)$ inclusive) it being understood that the column under $(5^2)$ is filled with 1's (from the partition $(5^2)$ to $(5,1^5)$); the correspondent to this column, namely $(4,5)$, being absent from the table for $n = 9$. In completing the table for $n = 10$ it is convenient, in dealing with a four or five element partition which

ends in a 1 or 2, to use the corresponding partition in the table for $n = 9$ or 8, respectively. E. g. to obtain the analysis of $\Delta(4, 3, 2, 1)$ we use, from the table for $n = 9$, the result

$$q_4 q_3 q_2 = \{9\} + 2\{8, 1\} + 3\{7, 2\} + \{7, 1^2\} + 3\{6, 3\} + 2\{6, 2, 1\} \\ + 2\{5, 4\} + 2\{5, 3, 1\} + \{5, 2^2\} + \{4^2, 1\} + \{4, 3, 2\}.$$

Hence

$$q_4 q_3 q_2 q_1 = \{9\}\{1\} + 2\{8, 1\}\{1\} \cdots$$

and from the theorem concerning the analysis of the direct product of irreducible representations, given in the following section, we have

$$\{9\}\{1\} = \{10\} + \{9, 1\}; \qquad \{8, 1\}\{1\} = \{9, 1\} + \{8, 2\} + \{8, 1^2\} \text{ etc.,}$$

so that on collecting we obtain

$$\Delta(4, 3, 2, 1) = D(10) + 3D(9, 1) + 5D(8, 2) + 3D(8, 1^2) + 6D(7, 3) \\ + 6D(7, 2, 1) + D(7, 1^3) + 5D(6, 4) + 7D(6, 3, 1) \\ + 3D(6, 2^2) + 2D(6, 2, 1^2) + 2D(5^2) + 5D(5, 4, 1) \\ + 4D(5, 3, 2) + 2D(5, 3, 1^2) + D(5, 2^2, 1) + 2D(4^2, 2) \\ + D(4^2, 1^2) + D(4, 3^2) + D(4, 3, 2, 1).$$

**Tables furnishing the analysis of $\Delta(\lambda)$ for values of $n$ from 2 to 9 inclusive.**

1. n = 2.

| | (2) | (1²) |
|---|---|---|
| Δ(2) | 1 | |
| Δ(1²) | 1 | 1 |

2. n = 3.

| | (3) | (2,1) | (1³) |
|---|---|---|---|
| Δ(3) | 1 | | |
| Δ(2,1) | 1 | 1 | |
| Δ(1³) | 1 | 2 | 1 |

3. n = 4.

| | (4) | (3,1) | (2²) | (2,1²) | (1⁴) |
|---|---|---|---|---|---|
| Δ(4) | 1 | | | | |
| Δ(3,1) | 1 | 1 | | | |
| Δ(2²) | 1 | 1 | 1 | | |
| Δ(2,1²) | 1 | 2 | 1 | 1 | |
| Δ(1⁴) | 1 | 3 | 2 | 3 | 1 |

4. n = 5.

| | (5) | (4,1) | (3,2) | (3,1²) | (2²,1) | (2,1³) | (1⁵) |
|---|---|---|---|---|---|---|---|
| Δ(5) | 1 | | | | | | |
| Δ(4,1) | 1 | 1 | | | | | |
| Δ(3,2) | 1 | 1 | 1 | | | | |
| Δ(3,1²) | 1 | 2 | 1 | 1 | | | |
| Δ(2²,1) | 1 | 2 | 2 | 1 | 1 | | |
| Δ(2,1³) | 1 | 3 | 3 | 3 | 2 | 1 | |
| Δ(1⁵) | 1 | 4 | 5 | 6 | 5 | 4 | 1 |

5. n = 6.

| | (6) | (5,1) | (4,2) | (4,1²) | (3²) | (3,2,1) | (3,1³) | (2³) | (2²,1²) | (2,1⁴) | (1⁶) |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Δ(6) | 1 | | | | | | | | | | |
| Δ(5,1) | 1 | 1 | | | | | | | | | |
| Δ(4,2) | 1 | 1 | 1 | | | | | | | | |
| Δ(4,1²) | 1 | 2 | 1 | 1 | | | | | | | |
| Δ(3²) | 1 | 1 | 1 | 0 | 1 | | | | | | |
| Δ(3,2,1) | 1 | 2 | 2 | 1 | 1 | 1 | | | | | |
| Δ(3,1³) | 1 | 3 | 3 | 3 | 1 | 2 | 1 | | | | |
| Δ(2³) | 1 | 2 | 3 | 1 | 1 | 2 | 0 | 1 | | | |
| Δ(2²,1²) | 1 | 3 | 4 | 3 | 2 | 4 | 1 | 1 | 1 | | |
| Δ(2,1⁴) | 1 | 4 | 6 | 6 | 3 | 8 | 4 | 2 | 3 | 1 | |
| Δ(1⁶) | 1 | 5 | 9 | 10 | 5 | 16 | 10 | 5 | 9 | 5 | 1 |

6. n=7.

| | (7) | (6,1) | (5,2) | $(5,1^2)$ | (4,3) | (4,2,1) | $(4,1^3)$ | $(3^2,1)$ | $(3,2^2)$ | $(3,2,1^2)$ | $(3,1^4)$ | $(2^3,1)$ | $(2^2,1^3)$ | $(2,1^5)$ | $(1^7)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| (7) | 1 | | | | | | | | | | | | | | |
| $\Delta(6,1)$ | 1 | 1 | | | | | | | | | | | | | |
| $\Delta(5,2)$ | 1 | 1 | 1 | | | | | | | | | | | | |
| $\Delta(5,1^2)$ | 1 | 2 | 1 | 1 | | | | | | | | | | | |
| $\Delta(4,3)$ | 1 | 1 | 1 | 0 | 1 | | | | | | | | | | |
| $\Delta(4,2,1)$ | 1 | 2 | 2 | 1 | 1 | 1 | | | | | | | | | |
| $\Delta(4,1^3)$ | 1 | 3 | 3 | 3 | 1 | 2 | 1 | | | | | | | | |
| $\Delta(3^2,1)$ | 1 | 2 | 2 | 1 | 2 | 1 | 0 | 1 | | | | | | | |
| $\Delta(3,2^2)$ | 1 | 2 | 3 | 1 | 2 | 2 | 0 | 1 | 1 | | | | | | |
| $\Delta(3,2,1^2)$ | 1 | 3 | 4 | 3 | 3 | 4 | 1 | 2 | 1 | 1 | | | | | |
| $\Delta(3,1^4)$ | 1 | 4 | 6 | 6 | 4 | 8 | 4 | 3 | 2 | 3 | 1 | | | | |
| $\Delta(2^3,1)$ | 1 | 3 | 5 | 3 | 4 | 6 | 1 | 3 | 3 | 2 | 0 | 1 | | | |
| $\Delta(2^2,1^3)$ | 1 | 4 | 7 | 6 | 6 | 11 | 4 | 6 | 5 | 6 | 1 | 2 | 1 | | |
| $\Delta(2,1^5)$ | 1 | 5 | 10 | 10 | 9 | 20 | 10 | 11 | 10 | 15 | 5 | 5 | 4 | 1 | |
| $\Delta(1^7)$ | 1 | 6 | 14 | 15 | 14 | 35 | 20 | 21 | 21 | 35 | 15 | 14 | 14 | 6 | 1 |

7. n=8.

| | (8) | (7,1) | (6,2) | $(6,1^2)$ | (5,3) | (5,2,1) | $(5,1^3)$ | $(4^2)$ | (4,3,1) | $(4,2^2)$ | $(4,2,1^2)$ | $(4,1^4)$ | $(3^2,2)$ | $(3^2,1^2)$ | $(3,2^2,1)$ | $(3,2,1^3)$ | $(3,1^5)$ | $(2^4)$ | $(2^3,1^2)$ | $(2^2,1^4)$ | $(2,1^6)$ | $(1^8)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\Delta(8)$ | 1 | | | | | | | | | | | | | | | | | | | | | |
| $\Delta(7,1)$ | 1 | 1 | | | | | | | | | | | | | | | | | | | | |
| $\Delta(6,2)$ | 1 | 1 | 1 | | | | | | | | | | | | | | | | | | | |
| $\Delta(6,1^2)$ | 1 | 2 | 1 | 1 | | | | | | | | | | | | | | | | | | |
| $\Delta(5,3)$ | 1 | 1 | 1 | 0 | 1 | | | | | | | | | | | | | | | | | |
| $\Delta(5,2,1)$ | 1 | 2 | 2 | 1 | 1 | 1 | | | | | | | | | | | | | | | | |
| $\Delta(5,1^3)$ | 1 | 3 | 3 | 3 | 1 | 2 | 1 | | | | | | | | | | | | | | | |
| $\Delta(4^2)$ | 1 | 1 | 1 | 0 | 1 | 0 | 0 | 1 | | | | | | | | | | | | | | |
| $\Delta(4,3,1)$ | 1 | 2 | 2 | 1 | 2 | 1 | 0 | 1 | 1 | | | | | | | | | | | | | |
| $\Delta(4,2^2)$ | 1 | 2 | 3 | 1 | 2 | 2 | 0 | 1 | 1 | 1 | | | | | | | | | | | | |
| $\Delta(4,2,1^2)$ | 1 | 3 | 4 | 3 | 3 | 4 | 1 | 1 | 2 | 1 | 1 | | | | | | | | | | | |
| $\Delta(4,1^4)$ | 1 | 4 | 6 | 6 | 4 | 8 | 4 | 1 | 3 | 2 | 3 | 1 | | | | | | | | | | |
| $\Delta(3^2,2)$ | 1 | 2 | 3 | 1 | 3 | 2 | 0 | 1 | 2 | 1 | 0 | 0 | 1 | | | | | | | | | |
| $\Delta(3^2,1^2)$ | 1 | 3 | 4 | 3 | 4 | 4 | 1 | 2 | 4 | 1 | 1 | 0 | 1 | 1 | | | | | | | | |
| $\Delta(3,2^2,1)$ | 1 | 3 | 5 | 3 | 5 | 6 | 1 | 2 | 5 | 3 | 2 | 0 | 2 | 1 | 1 | | | | | | | |
| $\Delta(3,2,1^3)$ | 1 | 4 | 7 | 6 | 7 | 11 | 4 | 3 | 9 | 5 | 6 | 1 | 3 | 3 | 2 | 1 | | | | | | |
| $\Delta(3,1^5)$ | 1 | 5 | 10 | 10 | 10 | 20 | 10 | 4 | 15 | 10 | 15 | 5 | 5 | 6 | 5 | 4 | 1 | | | | | |
| $\Delta(2^4)$ | 1 | 3 | 6 | 3 | 6 | 8 | 1 | 3 | 7 | 6 | 3 | 0 | 3 | 2 | 3 | 0 | 0 | 1 | | | | |
| $\Delta(2^3,1^2)$ | 1 | 4 | 8 | 6 | 9 | 14 | 4 | 4 | 13 | 9 | 9 | 1 | 6 | 5 | 6 | 2 | 0 | 1 | 1 | | | |
| $\Delta(2^2,1^4)$ | 1 | 5 | 11 | 10 | 13 | 24 | 10 | 6 | 23 | 16 | 21 | 5 | 11 | 12 | 13 | 8 | 1 | 2 | 3 | 1 | | |
| $\Delta(2,1^6)$ | 1 | 6 | 15 | 15 | 19 | 40 | 20 | 9 | 40 | 30 | 45 | 15 | 21 | 26 | 30 | 24 | 6 | 5 | 9 | 5 | 1 | |
| $\Delta(1^8)$ | 1 | 7 | 20 | 21 | 28 | 64 | 35 | 14 | 70 | 56 | 90 | 35 | 42 | 56 | 70 | 64 | 21 | 14 | 28 | 20 | 7 | 1 |

8. $n=9$.

| | $(9)$ | $(8,1)$ | $(7,2)$ | $(7,1^2)$ | $(6,3)$ | $(6,2,1)$ | $(6,1^3)$ | $(5,4)$ | $(5,3,1)$ | $(5,2^2)$ | $(5,2,1^2)$ | $(5,1^4)$ | $(4^2,1)$ | $(4,3,2)$ | $(4,3,1^2)$ | $(4,2^2,1)$ | $(4,2,1^3)$ | $(4,1^5)$ | $(3^3)$ | $(3^2,2,1)$ | $(3^2,1^3)$ | $(3,2^3)$ | $(3,2^2,1^2)$ | $(3,2,1^4)$ | $(3,1^6)$ | $(2^4,1)$ | $(2^3,1^3)$ | $(2^2,1^5)$ | $(2,1^7)$ | $(1^9)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\Delta(9)$ | 1 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| $\Delta(8,1)$ | 1 | 1 | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| $\Delta(7,2)$ | 1 | 1 | 1 | | | | | | | | | | | | | | | | | | | | | | | | | | | |
| $\Delta(7,1^2)$ | 1 | 2 | 1 | 1 | | | | | | | | | | | | | | | | | | | | | | | | | | |
| $\Delta(6,3)$ | 1 | 1 | 1 | 0 | 1 | | | | | | | | | | | | | | | | | | | | | | | | | |
| $\Delta(6,2,1)$ | 1 | 2 | 2 | 1 | 1 | 1 | | | | | | | | | | | | | | | | | | | | | | | | |
| $\Delta(6,1^3)$ | 1 | 3 | 3 | 3 | 1 | 2 | 1 | | | | | | | | | | | | | | | | | | | | | | | |
| $\Delta(5,4)$ | 1 | 1 | 1 | 0 | 1 | 0 | 0 | 1 | | | | | | | | | | | | | | | | | | | | | | |
| $\Delta(5,3,1)$ | 1 | 2 | 2 | 1 | 2 | 1 | 0 | 1 | 1 | | | | | | | | | | | | | | | | | | | | | |
| $\Delta(5,2^2)$ | 1 | 2 | 3 | 1 | 2 | 2 | 0 | 1 | 1 | 1 | | | | | | | | | | | | | | | | | | | | |
| $\Delta(5,2,1^2)$ | 1 | 3 | 4 | 3 | 3 | 4 | 1 | 1 | 2 | 1 | 1 | | | | | | | | | | | | | | | | | | | |
| $\Delta(5,1^4)$ | 1 | 4 | 6 | 6 | 4 | 8 | 4 | 1 | 3 | 2 | 3 | 1 | | | | | | | | | | | | | | | | | | |
| $\Delta(4^2,1)$ | 1 | 2 | 2 | 1 | 2 | 1 | 0 | 2 | 1 | 0 | 0 | 0 | 1 | | | | | | | | | | | | | | | | | |
| $\Delta(4,3,2)$ | 1 | 2 | 3 | 1 | 3 | 2 | 0 | 2 | 2 | 1 | 0 | 0 | 1 | 1 | | | | | | | | | | | | | | | | |
| $\Delta(4,3,1^2)$ | 1 | 3 | 4 | 3 | 4 | 4 | 1 | 3 | 4 | 1 | 1 | 0 | 2 | 1 | 1 | | | | | | | | | | | | | | | |
| $\Delta(4,2^2,1)$ | 1 | 3 | 5 | 3 | 5 | 6 | 1 | 3 | 5 | 3 | 2 | 0 | 2 | 2 | 1 | 1 | | | | | | | | | | | | | | |
| $\Delta(4,2,1^3)$ | 1 | 4 | 7 | 6 | 7 | 11 | 4 | 4 | 9 | 5 | 6 | 1 | 3 | 3 | 3 | 2 | 1 | | | | | | | | | | | | | |
| $\Delta(4,1^5)$ | 1 | 5 | 10 | 10 | 10 | 20 | 10 | 5 | 15 | 10 | 15 | 5 | 4 | 5 | 6 | 5 | 4 | 1 | | | | | | | | | | | | |
| $\Delta(3^3)$ | 1 | 2 | 3 | 1 | 4 | 2 | 0 | 2 | 3 | 1 | 0 | 0 | 1 | 2 | 0 | 0 | 0 | 0 | 1 | | | | | | | | | | | |
| $\Delta(3^2,2,1)$ | 1 | 3 | 5 | 3 | 6 | 6 | 1 | 4 | 7 | 3 | 2 | 0 | 3 | 4 | 2 | 1 | 0 | 0 | 1 | 1 | | | | | | | | | | |
| $\Delta(3^2,1^3)$ | 1 | 4 | 7 | 6 | 8 | 11 | 4 | 6 | 12 | 5 | 6 | 1 | 6 | 6 | 6 | 2 | 1 | 0 | 1 | 2 | 1 | | | | | | | | | |
| $\Delta(3,2^3)$ | 1 | 3 | 6 | 3 | 7 | 8 | 1 | 5 | 9 | 6 | 3 | 0 | 4 | 6 | 3 | 3 | 0 | 0 | 1 | 2 | 0 | 1 | | | | | | | | |
| $\Delta(3,2^2,1^2)$ | 1 | 4 | 8 | 6 | 10 | 14 | 4 | 7 | 16 | 9 | 9 | 1 | 7 | 10 | 8 | 6 | 2 | 0 | 2 | 4 | 1 | 1 | 1 | | | | | | | |
| $\Delta(3,2,1^4)$ | 1 | 5 | 11 | 10 | 14 | 24 | 10 | 10 | 27 | 16 | 21 | 5 | 12 | 17 | 18 | 13 | 8 | 1 | 3 | 8 | 4 | 2 | 3 | 1 | | | | | | |
| $\Delta(3,1^6)$ | 1 | 6 | 15 | 15 | 20 | 40 | 20 | 14 | 45 | 30 | 45 | 15 | 19 | 30 | 36 | 30 | 24 | 6 | 5 | 16 | 10 | 5 | 9 | 5 | 1 | | | | | |
| $\Delta(2^4,1)$ | 1 | 4 | 9 | 6 | 12 | 17 | 4 | 9 | 21 | 14 | 12 | 1 | 10 | 16 | 12 | 12 | 3 | 0 | 3 | 8 | 2 | 4 | 3 | 0 | 0 | 1 | | | | |
| $\Delta(2^3,1^3)$ | 1 | 5 | 12 | 10 | 17 | 28 | 10 | 13 | 36 | 23 | 27 | 5 | 17 | 28 | 27 | 24 | 12 | 1 | 6 | 17 | 7 | 7 | 9 | 2 | 0 | 2 | 1 | | | |
| $\Delta(2^2,1^5)$ | 1 | 6 | 16 | 15 | 24 | 45 | 20 | 19 | 60 | 40 | 55 | 15 | 29 | 50 | 56 | 50 | 34 | 6 | 11 | 36 | 20 | 15 | 24 | 10 | 1 | 5 | 4 | 1 | | |
| $\Delta(2,1^7)$ | 1 | 7 | 21 | 21 | 34 | 70 | 35 | 28 | 99 | 70 | 105 | 35 | 49 | 91 | 111 | 105 | 84 | 21 | 21 | 77 | 50 | 35 | 63 | 35 | 7 | 14 | 14 | 6 | 1 | |
| $\Delta(1^9)$ | 1 | 8 | 27 | 28 | 48 | 105 | 56 | 42 | 162 | 120 | 189 | 70 | 84 | 168 | 216 | 216 | 189 | 56 | 42 | 168 | 120 | 84 | 162 | 105 | 28 | 42 | 48 | 27 | 8 | 1 |

**The direct product of irreducible representations.** We consider two sets of $n$ and $m$ letters, neither set having a common letter so that the number of distinct letters in the two sets taken together is $n+m$. If $(\lambda)$ is an arbitrary partition of $n$ and $(\mu)$ an arbitrary partition of $m$ and $D(\lambda)$, $D(\mu)$ the attached irreducible representations of the symmetric groups on $n$ and $m$ letters, respectively, the direct product $D(\lambda)\cdot D(\mu)$ is a representation, in general reducible, of the symmetric group on $n+m$ letters whose characteristic is the product $\{\lambda\}\{\mu\}$ of the characteristics of $D(\lambda)$ and $D(\mu)$. Our problem is the analysis of $D(\lambda)\cdot D(\mu)$ into its irreducible components. For the sake of brevity we shall omit the symbols $D$ and write $(\lambda)\cdot(\mu)$ for $D(\lambda)\cdot D(\mu)$. If $(\nu)$ is a typical partition of $n+m$ a relation $(\lambda)\cdot(\mu)=\sum_{(\nu)} c_{(\nu)}(\nu)$ implies $\{\lambda\}\{\mu\}=\sum_{(\nu)} c_{(\nu)}\{\nu\}$ since $\{\lambda\}\{\mu\}$ is the characteristic of $(\lambda)\cdot(\mu)$. In order to arrive at a solution of our problem we first remark that the fundamental recurrence formula (13) may be generalised as follows. Let $\xi_j$ be an operator which decreases the $j$-th member $\lambda_j$ of the partition

$$(\lambda)=(\lambda_1,\cdot\cdot\cdot,\lambda_k)=(\lambda_1,\cdot\cdot\cdot,\lambda_n)$$

by unity $(j=1,\cdot\cdot\cdot,n)$. Then

$$\xi_j{}^p\{\lambda_1,\cdot\cdot\cdot,\lambda_k\}=\xi_j{}^p\{\lambda_1,\cdot\cdot\cdot,\lambda_n\}=\{\lambda_1,\cdot\cdot\cdot,\lambda_j-p,\cdot\cdot\cdot,\lambda_n\}$$

so that $\xi_j{}^p\{\lambda_1,\cdot\cdot\cdot,\lambda_k\}=0$ if $j>k$ for then $\{\lambda_1,\cdot\cdot\cdot,\lambda_j-p,\cdot\cdot\cdot,\lambda_n\}$ ends in a negative integer after the zeros at the end have been discarded. On writing

$S_p=\xi_1{}^p+\cdot\cdot\cdot+\xi_n{}^p$ we have

$$S_p\{\lambda_1,\cdot\cdot\cdot,\lambda_k\}=\{\lambda_1-p,\cdot\cdot\cdot,\lambda_k\}+\cdot\cdot\cdot+\{\lambda_1,\cdot\cdot\cdot,\lambda_k-p\}$$

so that our formula (13) may be written in the form

$$\{\lambda_1,\cdot\cdot\cdot,\lambda_k\}_{(\alpha)}=S_p\{\lambda_1,\cdot\cdot\cdot,\lambda_k\}_{(\alpha')}$$

and we may say that we have stripped off one cycle of $p$ letters from $(\alpha)$. Following this by stripping from $(\alpha')$ a cycle of $q$ letters we obtain

$$\{\lambda_1,\cdot\cdot\cdot,\lambda_k\}_\alpha=S_qS_p\{\lambda_1,\cdot\cdot\cdot,\lambda_k\}_{(\alpha'')}$$

where $(\alpha'')$ is the class, of the symmetric group on $n-p-q$ letters, which contains one less cycle on $p$ letters and one less cycle on $q$ letters than the class $(\alpha)$ of the symmetric group on $n$ letters. More generally we may strip

off $\beta_1$ unary cycles, $\beta_2$ binary cycles etc.; to write the corresponding generalisation of the recurrence relation (13) it is a little more convenient to change the notation slightly so that $n$ is replaced by $n + m$. Then $(\alpha)$ is a class of $n + m$ and we strip off $\beta_1$ unary cycles, $\beta_2$ binary cycles, $\cdots \beta_n$ $n$-ary cycles where $(\beta)$ is a class of $n$. Denoting by $(\gamma)$ the class of $m$ which is such that $(\beta) + (\gamma) = (\alpha)$ our generalised recurrence relation appears in the form

$$\{\lambda\}_{(\alpha)} = S_1^{\beta_1} \cdots S_n^{\beta_n}\{\lambda\}_{(\gamma)}.$$

By means of (3) this may be written in the form

$$\{\lambda\}_{(\alpha)} = \sum_{(\theta)} \chi_{(\theta)}{}^{(\beta)}\phi_\theta(S)\{\lambda\}_{(\gamma)} \tag{19}$$

where the summation is over the partitions $(\theta)$ of $n$ and $(\beta)$, $(\gamma)$ are any classes of $n$ and $m$, respectively, whose sum is the class $(\alpha)$ of $n + m$.

Let now

$$\phi_{(\epsilon)}(s) = \frac{1}{n!}\sum_{(\delta)} N_{(\delta)}\chi_{(\epsilon)}{}^{(\delta)}s^{(\delta)}$$

be any simple characteristic of the symmetric group on $n$ letters (so that $(\epsilon)$ is a partition of $n$) and, similarly, let

$$\phi_{(\nu)}(s) = \frac{1}{m!}\sum_{(\tau)} M_{(\tau)}\chi_{(\nu)}{}^{(\tau)}s^{(\tau)}$$

be any simple characteristic of the symmetric group on $m$ letters, $(\nu)$ being a partition of $m$; their product is

$$\phi_{(\epsilon)}(s)\phi_{(\nu)}(s) = \frac{1}{n!}\,\frac{1}{m!}\sum_{(\delta)(\tau)} N_{(\delta)}M_{(\tau)}\chi_{(\epsilon)}{}^{(\delta)}\chi_{(\nu)}{}^{(\tau)}s^{(\delta)+(\tau)}$$

and since $(\delta) + (\tau)$ is a partition of $n + m$ we have, from (3),

$$s^{(\delta)+(\tau)} = \sum_{(\alpha)} \chi_{(\alpha)}{}^{(\delta)+(\tau)}\phi_{(\alpha)}(s)$$

the summation being over all partitions $(\alpha)$ of $n + m$. On substituting for $\chi_{(\alpha)}{}^{(\delta)+(\tau)}$ its value $\sum_\theta \chi_{(\theta)}{}^{(\delta)}\{\phi_\theta(S)\chi_\alpha\}^{(\tau)}$ from (19) and summing with respect to $(\delta)$ we obtain, in view of the orthogonality relations (1) between the characters of the symmetric group on $n$ letters,

$$\phi_{(\epsilon)}(s)\phi_{(\nu)}(s) = \frac{1}{m!}\sum_{(\alpha)(\tau)} M_{(\tau)}\chi_{(\nu)}{}^{(\tau)}\{\phi_{(\epsilon)}(S)\chi_{(\alpha)}\}^{(\tau)}\phi_{(\alpha)}(s) \cdots. \tag{20}$$

Now $\phi_{(\epsilon)}(S)$ is a symmetric function, of degree $n$, with integral coefficients, of the $n+m$ operators $\xi_j$ and so is of the form $\sum_{(\pi)} c_\pi[\xi_1^{\pi_1}\cdots\xi_n^{\pi_n}]$ where $(\pi)$ is a partition of $n$ and $[\xi_1^{\pi_1}\xi_2^{\pi_2}\cdots\xi_n^{\pi_n}]$ denotes the symmetric function of the $n+m$ operators $(\xi_1,\cdots,\xi_{n+m})$ whose leading term is $\xi_1^{\pi_1}\xi_2^{\pi_2}\cdots\xi_n^{\pi_n}$. The result of operating with $\xi_1^{\pi_1}\cdots\xi_n^{\pi_n}$ on $\chi_{(\alpha)}$ is

$$\chi_{(\alpha_1-\pi_1,\alpha_2-\pi_2,\ldots,\alpha_n-\pi_n,\alpha_{n+1},\ldots,\alpha_{n+m})}$$

and we may denote the result of operating with $[\xi_1^{\pi_1}\cdots\xi_n^{\pi_n}]$ on $\chi_{(\alpha)}$ by $\chi_{[(\alpha)-(\pi)]}$. The summation with respect to $\tau$ yields zero, owing to the orthogonality relations between the characters of the symmetric group on $m$ letters, save when $(\alpha)$ is such that one member of $[(\alpha)-(\pi)]$ is the same as $(\nu)$, with the same convention as before regarding the rearrangement of disordered partitions; in which case the coefficient of $\phi_{(\alpha)}(s)$ is $c_{(\pi)}$. The simplest examples may serve to make the theory clear; thus let $(\epsilon)=(1)$ so that we wish to analyse $\{1\}\{\nu_1,\cdots,\nu_k\}$; the only $\phi_{(\alpha)}(s)$ which appear in this product are those for which $(\alpha)$ is obtained from $(\nu_1,\cdots,\nu_k,0)$ by adding unity to one of its members and these all occur with coefficient unity; for

$$\phi_{(1)}(S)=S_1=\xi_1+\xi_2+\cdots\xi_{n+1}.$$

Hence

$$\{1\}\{\nu_1,\cdots,\nu_k\}=\{\nu_1+1,\nu_2,\cdots,\nu_k\}+\cdots+\{\nu_1,\cdots,\nu_k+1\}+\{\nu_1,\cdots,\nu_k,1\}.$$

E. g.,
$$\begin{aligned}\{1\}\{4,2^2\}&=\{5,2^2\}+\{4,3,2\}+\{4,2,3\}+\{4,2^2,1\}\\&=\{5,2^2\}+\{4,3,2\}+\{4,2^2,1\}\end{aligned}$$

or, equivalently $(1)\cdot(4,2^2)=(5,2^2)+(4,3,2)+(4,2^2,1)$. The next simplest example is furnished by $\{2\}\{\nu_1,\cdots,\nu_k\}$; here

$$\phi_2(S)=p_2(\xi)=\Sigma\xi_j^2+\Sigma\xi_j\xi_p$$

and so

$$\begin{aligned}\{2\}\{\nu_1,\cdots,\nu_k\}&=\{\nu_1+2,\cdots,\nu_k\}+\cdots+\{\nu_1,\cdots,\nu_k,2\}\\&\quad+\{\nu_1+1,\nu_2+1,\cdots,\nu_k\}+\cdots\\&\quad+\{\nu_1+1,\cdots,\nu_k,1\}+\cdots\end{aligned}$$

the terms $\{\nu_1+1,\cdots,\nu_k,0,1\}$ vanishing and the terms $\{\nu_1,\cdots,\nu_k,0,2\}$ and $\{\nu_1,\cdots,\nu_k,1,1\}$ cancelling one another.

E. g.,
$$\begin{aligned}\{2\}\{3,2\}&=\{5,2\}+\{3,4\}+\{3,2^2\}+\{4,3\}+\{4,2,1\}+\{3^2,1\}\\&=\{5,2\}+\{4,3\}+\{4,2,1\}+\{3^2,1\}+\{3,2^2\}.\end{aligned}$$

Similarly since

$$\phi_3(S) = p_3(\xi) = \Sigma\xi_j{}^3 + \Sigma\xi_j{}^2\xi_l + \Sigma\xi_j\xi_l\xi_m$$

we have

$$\begin{aligned}\{3\}\{\nu_1,\cdots,\nu_k\} = \{&\nu_1+3,\cdots,\nu_k\}+\cdots+\{\nu_1,\cdots,\nu_k,3\}\\ &+\{\nu_1+2,\nu_2+1,\cdots,\nu_k\}+\cdots\\ &+\{\nu_1+2,\cdots,\nu_k,1\}+\cdots\\ &+\{\nu_1+1,\cdots,\nu_k,2\}+\cdots\\ &+\{\nu_1+1,\nu_2+1,\nu_3+1,\cdots,\nu_k\}+\cdots\\ &+\{\nu_1+1,\nu_2+1,\cdots,\nu_k,1\}+\cdots.\end{aligned}$$

(the remaining terms vanishing or cancelling each other).

E. g.,
$$\begin{aligned}\{3\}\{2,1^2\} &= \{5,1^2\}+\{2,4,1\}+\{2,1,4\}+\{2,1^2,3\}\\ &+\{4,2,1\}+\{3^2,1\}+\{4,1,2\}\\ &\qquad+\{3,1,3\}+\{2,3,2\}+\{2^2,3\}\\ &+\{4,1^3\}+\{2,3,1^2\}+\{2,1,3,1\}\\ &\qquad+\{3,1^2,2\}+\{2^2,1,2\}+\{2,1,2^2\}\\ &+\{3,2^2\}+\{3,2,1^2\}+\{3,1,2,1\}+\{2^3,1\}\\ &=\{5,1^2\}+\{4,2,1\}+\{4,1^3\}+\{3,2,1^2\}.\end{aligned}$$

Since $\phi_{(1^2)}(S) = \sigma_2(\xi) = \Sigma\xi_j\xi_l$ we have

$$\begin{aligned}\{1^2\}\{\nu,\cdots,\nu_k\} = \{&\nu_1+1,\nu_2+1,\cdots,\nu_k\}\\ &+\{\nu_1+1,\cdots,\nu_k,1\}+\cdots+\{\nu_1,\cdots,\nu_k,1,1\}\end{aligned}$$

E. g.,
$$\begin{aligned}\{1^2\}\{3,2,1\} &= \{4,3,1\}+\{4,2^2\}+\{3^2,2\}\\ &+\{4,2,1^2\}+\{3^2,1^2\}+\{3,2^2,1\}+\{3,2,1^3\}.\end{aligned}$$

It is clear that whilst this method is entirely practicable when one of the factors is $\{2\}$, $\{3\}$, $\{1^2\}$, $\{1^3\}$ it rapidly becomes very tedious in other cases. We give below tables of all direct products $(\lambda)\cdot(\mu)$ for which $n+m\leq 9$ and we found the following method, which is sufficiently illustrated by an example, entirely convenient. Suppose we wish $\{3,1\}\{2^2\}$; we write $\{3,1\}=\{3\}\{1\}-\{4\}$ (since $\begin{vmatrix} q_3 & q_4\\ q_0 & q_1\end{vmatrix} = q_3q_1 - q_4$) and we see that the calculation rests on that of $\{4\}\{2^2\}$. But $\{4,2^2\} = \{4\}\{2^2\} - \{1\}\{5,2\} + \{5,3\}$ since

$$\begin{vmatrix} q_4 & q_5 & q_6\\ q_1 & q_2 & q_3\\ q_0 & q_1 & q_2\end{vmatrix} = q_4\begin{vmatrix} q_2 & q_3\\ q_1 & q_2\end{vmatrix} - q_1\begin{vmatrix} q_5 & q_6\\ q_1 & q_2\end{vmatrix} + \begin{vmatrix} q_5 & q_6\\ q_2 & q_3\end{vmatrix}.$$

Hence

$$\{4\}\{2^2\} = \{4, 2^2\} + \{1\}\{5, 2\} - \{5, 3\}$$
$$= \{6, 2\} + \{5, 2, 1\} + \{4, 2^2\}.$$

Similarly

$$\{3\}\{2^2\} = \{5, 2\} + \{4, 2, 1\} + \{3, 2^2\}$$

and so

$$\begin{aligned}\{1\}\{3\}\{2\}^2 = & [\{6, 2\} + \{5, 3\} + \{5, 2, 1\}] \\ & + [\{5, 2, 1\} + \{4, 3, 1\} + \{4, 2^2\} + \{4, 2, 1^2\}] \\ & + [\{4, 2^2\} + \{3^2, 2\} + \{3, 2, 3\} + \{3, 2^2, 1\}] \\ = & \{6, 2\} + \{5, 3\} + 2\{5, 2, 1\} + \{4, 3, 1\} + 2\{4, 2^2\} + \{4, 2, 1^2\} \\ & + \{3^2, 2\} + \{3, 2^2, 1\}.\end{aligned}$$

Hence

$$\begin{aligned}\{3, 1\}\{2^2\} = & \{5, 3\} + \{5, 2, 1\} \\ & + \{4, 3, 1\} + \{4, 2^2\} + \{4, 2, 1^2\} + \{3^2, 2\} + \{3, 2^2, 1\}.\end{aligned}$$

In the following tables the irreducible representations are written across the top and the desired direct products are indicated down the left. As examples of how the tables are read we cite the following:

$$n + m = 3;\ (2) \cdot (1) = (3) + (2, 1)$$
$$n + m = 4;\ (2, 1) \cdot (1) = (3, 1) + (2^2) + (2, 1^2)$$
$$n + m = 5;\ (3) \cdot (1^2) = (4, 1) + (3, 1^2)$$
$$\begin{aligned}n + m = 6;\ (2, 1) \cdot (2, 1) = & (4, 2) + (4, 1^2) + (3^2) \\ & + 2(3, 2, 1) + (2^3) + (3, 1^3) + (2^2, 1^2).\end{aligned}$$

Since a change of sign of $(s_2, s_4, \cdots)$ sends $\{\lambda\}$ into the associated characteristic $\{\mu\}$ it is clear that $\{\mu\}\{\mu'\}$ is obtained from $\{\lambda\}\{\lambda'\}$ by merely taking the associated characteristics or representations; e.g. from $\{3\}\{1^2\} = \{4, 1\} + \{3, 1^2\}$ we read $\{1^3\}\{2\} = \{2, 1^3\} + \{3, 1^2\}$. We use this trivially evident fact to materially cut down the size of the tables (without causing trouble to the user) by writing $\{\mu\}\{\mu'\}$ on the right side of the table directly opposite $\{\lambda\}\{\lambda'\}$— where $\{\mu\}$ and $\{\lambda\}$ are associated simple characteristics of the symmetric group on $n$ letters whilst $\{\mu'\}$ and $\{\lambda'\}$ are associated simple characteristics of the symmetric group on $m$ letters. It being understood that when we pick up our direct product on the right-hand side of the table we find the irreducible representations of the symmetric group on $n + m$ letters which occur in the

analysis of the direct product at the *bottom* of the table; whilst when we pick up the direct product on the left-hand side of the table we find the representations which occur in its analysis at the *top* of the table. For convenience of printing, the tables for $n + m = 8, 9$ have been turned so that the top of the page is the right-hand side of the table and the left-hand side of the page is the top of the table.

**Tables furnishing the analysis of the direct product $(\lambda) \cdot (\lambda')$ for all values of $n + m$ from 2 to 9 inclusive.**

1. n+m=2.

$$(1).(1) = (2) + (1^2)$$

2. n+m=3.

| | $(3)$ | $(2,1)$ | $(1^3)$ | |
|---|---|---|---|---|
| $(2).(1)$ | 1 | 1 | | $(1^2).(1)$ |
| | $(1^3)$ | $(2,1)$ | $(3)$ | |

3. n+m=4.

| | $(4)$ | $(3,1)$ | $(2^2)$ | $(2,1^2)$ | $(1^4)$ | |
|---|---|---|---|---|---|---|
| $(3).(1)$ | 1 | 1 | | | | $(1^3).(1)$ |
| $(2,1).(1)$ | | 1 | 1 | 1 | | $(2,1).(1)$ |
| $(2).(2)$ | 1 | 1 | 1 | | | $(1^2).(1^2)$ |
| $(2).(1^2)$ | | 1 | | 1 | | $(1^2).(2)$ |
| | $(1^4)$ | $(2,1^2)$ | $(2^2)$ | $(3,1)$ | $(4)$ | |

4. n+m=5.

| | $(5)$ | $(4,1)$ | $(3,2)$ | $(3,1^2)$ | $(2^2,1)$ | $(2,1^3)$ | $(1^5)$ | |
|---|---|---|---|---|---|---|---|---|
| $(4).(1)$ | 1 | 1 | | | | | | $(1^4).(1)$ |
| $(3,1).(1)$ | | 1 | 1 | 1 | | | | $(2,1^2).(1)$ |
| $(2^2).(1)$ | | | 1 | | 1 | | | $(2^2).(1)$ |
| $(3).(2)$ | 1 | 1 | 1 | | | | | $(1^3).(1^2)$ |
| $(2,1).(2)$ | | 1 | 1 | 1 | 1 | | | $(2,1).(1^2)$ |
| $(1^3).(2)$ | | | | 1 | | 1 | | $(3).(1^2)$ |
| | $(1^5)$ | $(2,1^3)$ | $(2^2,1)$ | $(3,1^2)$ | $(3,2)$ | $(4,1)$ | $(5)$ | |

5. n+m=6.

| | (6) | (5,1) | (4,2) | $(4,1^2)$ | $(3)^2$ | (3,2,1) | $(3,1^3)$ | $(2^3)$ | $(2^2,1^2)$ | $(2,1^4)$ | $(1^6)$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| (5).(1) | 1 | 1 | | | | | | | | | | $(1^5)$.(1) |
| (4,1).(1) | | 1 | 1 | 1 | | | | | | | | $(2,1^3)$.(1) |
| (3,2).(1) | | | 1 | | 1 | 1 | | | | | | $(2^2,1)$.(1) |
| $(3,1^2)$.(1) | | | | 1 | | 1 | 1 | | | | | $(3,1^2)$.(1) |
| (4).(2) | 1 | 1 | 1 | | | | | | | | | $(1^4).(1^2)$ |
| (3,1).(2) | | 1 | 1 | 1 | 1 | 1 | | | | | | $(2,1^2).(1^2)$ |
| $(2^2)$.(2) | | | 1 | | | 1 | | 1 | | | | $(2^2).(1^2)$ |
| $(2,1^2)$.(2) | | | | 1 | | 1 | 1 | | 1 | | | $(3,1).(1^2)$ |
| $(1^4)$.(2) | | | | | | | 1 | | | 1 | | $(4).(1^2)$ |
| (3).(3) | 1 | 1 | 1 | | 1 | | | | | | | $(1^3).(1^3)$ |
| (3).(2,1) | | 1 | 1 | 1 | | 1 | | | | | | $(1^3)$.(2,1) |
| (3).$(1^3)$ | | | | 1 | | | 1 | | | | | $(1^3)$.(3) |
| (2,1).(2,1) | | | 1 | 1 | 1 | 2 | 1 | 1 | 1 | | | (2,1).(2,1) |
| | $(1^6)$ | $(2,1^4)$ | $(2^2,1^2)$ | $(3,1^3)$ | $(2^3)$ | (3,2,1) | $(4,1^2)$ | $(3^2)$ | (4,2) | (5,1) | (6) | |

6. n+m=7.

| | (7) | (6,1) | (5,2) | $(5,1^2)$ | (4,3) | (4,2,1) | $(4,1^3)$ | $(3^2,1)$ | $(3,2^2)$ | $(3,2,1^2)$ | $(3,1^4)$ | $(2^3,1)$ | $(2^2,1^3)$ | $(2,1^5)$ | $(1^7)$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| (6).(1) | 1 | 1 | | | | | | | | | | | | | | $(1^6)$.(1) |
| (5,1).(1) | | 1 | 1 | 1 | | | | | | | | | | | | $(2,1^4)$.(1) |
| (4,2).(1) | | | 1 | | 1 | 1 | | | | | | | | | | $(2^2,1^2)$.(1) |
| $(4,1^2)$.(1) | | | | 1 | | 1 | 1 | | | | | | | | | $(3,1^3)$.(1) |
| $(3^2)$.(1) | | | | | 1 | | | 1 | | | | | | | | $(2^3)$.(1) |
| (3,2,1).(1) | | | | | | 1 | | 1 | 1 | 1 | | | | | | (3,2,1).(1) |
| (5).(2) | 1 | 1 | 1 | | | | | | | | | | | | | $(1^5).(1^2)$ |
| (4,1).(2) | | 1 | 1 | 1 | 1 | 1 | | | | | | | | | | $(2,1^3).(1^2)$ |
| (3,2).(2) | | | 1 | | 1 | 1 | | 1 | 1 | | | | | | | $(2^2,1).(1^2)$ |
| $(3,1^2)$.(2) | | | | 1 | | 1 | 1 | 1 | | 1 | | | | | | $(3,1^2).(1^2)$ |
| $(2^2,1)$.(2) | | | | | | 1 | | | 1 | 1 | | 1 | | | | $(3,2).(1^2)$ |
| $(2,1^3)$.(2) | | | | | | | 1 | | | 1 | 1 | | 1 | | | $(4,1).(1^2)$ |
| $(1^5)$.(2) | | | | | | | | | | | 1 | | | 1 | | $(5).(1^2)$ |
| (4).(3) | 1 | 1 | 1 | | 1 | | | | | | | | | | | $(1^4).(1^3)$ |
| (3,1).(3) | | 1 | 1 | 1 | 1 | 1 | | 1 | | | | | | | | $(2,1^2).(1^3)$ |
| $(2^2)$.(3) | | | 1 | | | 1 | | | 1 | | | | | | | $(2^2).(1^3)$ |
| $(2,1^2)$.(3) | | | | 1 | | 1 | 1 | | | 1 | | | | | | $(3,1).(1^3)$ |
| $(1^4)$.(3) | | | | | | | 1 | | | | 1 | | | | | $(4).(1^3)$ |
| (4).(2,1) | | 1 | 1 | 1 | | 1 | | | | | | | | | | $(1^4)$.(2,1) |
| (3,1).(2,1) | | | 1 | 1 | 1 | 2 | 1 | 1 | 1 | 1 | | | | | | $(2,1^2)$.(2,1) |
| $(2^2)$.(2,1) | | | | | 1 | 1 | | 1 | 1 | 1 | | 1 | | | | $(2^2)$.(2,1) |
| | $(1^7)$ | $(2,1^5)$ | $(2^2,1^3)$ | $(3,1^4)$ | $(2^3,1)$ | $(3,2,1^2)$ | $(4,1^3)$ | $(3,2^2)$ | $(3^2,1)$ | (4,2,1) | $(5,1^2)$ | (4,3) | (5,2) | (6,1) | (7) | |

| 7. n+m=8 | $(8)$ | $(7,1)$ | $(6,2)$ | $(6,1^2)$ | $(5,3)$ | $(5,2,1)$ | $(5,1^3)$ | $(4^2)$ | $(4,3,1)$ | $(4,2^2)$ | $(4,2,1^2)$ | $(4,1^4)$ | $(3^2,2)$ | $(3^2,1^2)$ | $(3,2^2,1)$ | $(3,2,1^3)$ | $(3,1^5)$ | $(2^4)$ | $(2^3,1^2)$ | $(2^2,1^4)$ | $(2,1^6)$ | $(1^8)$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $(7).(1)$ | 1 | 1 | | | | | | | | | | | | | | | | | | | | | $(1^7).(1)$ |
| $(6,1).(1)$ | | 1 | 1 | 1 | | | | | | | | | | | | | | | | | | | $(2,1^5).(1)$ |
| $(5,2).(1)$ | | | 1 | | 1 | 1 | | | | | | | | | | | | | | | | | $(2^2,1^3).(1)$ |
| $(5,1^2).(1)$ | | | | 1 | | 1 | 1 | | | | | | | | | | | | | | | | $(3,1^4).(1)$ |
| $(4,3).(1)$ | | | | | 1 | | | 1 | 1 | | | | | | | | | | | | | | $(2^3,1).(1)$ |
| $(4,2,1).(1)$ | | | | | | 1 | | | 1 | 1 | 1 | | | | | | | | | | | | $(3,2,1^2).(1)$ |
| $(4,1^3).(1)$ | | | | | | | 1 | | | | 1 | 1 | | | | | | | | | | | $(4,1^3).(1)$ |
| $(3^2,1).(1)$ | | | | | | | | | 1 | | | | 1 | 1 | | | | | | | | | $(3,2^2).(1)$ |
| $(6).(2)$ | 1 | 1 | 1 | | | | | | | | | | | | | | | | | | | | $(1^6).(1^2)$ |
| $(5,1).(2)$ | | 1 | 1 | 1 | 1 | 1 | | | | | | | | | | | | | | | | | $(2,1^4).(1^2)$ |
| $(4,2).(2)$ | | | 1 | | 1 | 1 | | 1 | 1 | 1 | | | | | | | | | | | | | $(2^2,1^2).(1^2)$ |
| $(4,1^2).(2)$ | | | | 1 | | 1 | 1 | | 1 | | 1 | | | | | | | | | | | | $(3,1^3).(1^2)$ |
| $(3^2).(2)$ | | | | | 1 | | | | 1 | | | | 1 | | | | | | | | | | $(2^3).(1^2)$ |
| $(3,2,1).(2)$ | | | | | | 1 | | | 1 | 1 | 1 | | 1 | 1 | 1 | | | | | | | | $(3,2,1).(1^2)$ |
| $(3,1^3).(2)$ | | | | | | | 1 | | | | 1 | 1 | | 1 | | 1 | | | | | | | $(4,1^2).(1^2)$ |
| $(2^3).(2)$ | | | | | | | | | | 1 | | | | | 1 | | | 1 | | | | | $(3^2).(1^2)$ |
| $(2^2,1^2).(2)$ | | | | | | | | | | | 1 | | | | 1 | 1 | | | 1 | | | | $(4,2).(1^2)$ |
| $(2,1^4).(2)$ | | | | | | | | | | | | 1 | | | | 1 | 1 | | | 1 | | | $(5,1).(1^2)$ |
| $(1^6).(2)$ | | | | | | | | | | | | | | | | | 1 | | | | 1 | | $(6).(1^2)$ |
| $(5).(3)$ | 1 | 1 | 1 | | 1 | | | | | | | | | | | | | | | | | | $(1^5).(1^3)$ |
| $(4,1).(3)$ | | 1 | 1 | 1 | 1 | 1 | | 1 | 1 | | | | | | | | | | | | | | $(2,1^3).(1^3)$ |
| $(3,2).(3)$ | | | 1 | | 1 | 1 | | | 1 | 1 | | | 1 | | | | | | | | | | $(2^2,1).(1^3)$ |
| $(3,1^2).(3)$ | | | | 1 | | 1 | 1 | | 1 | | 1 | | | 1 | | | | | | | | | $(3,1^2).(1^3)$ |
| $(2^2,1).(3)$ | | | | | | 1 | | | | 1 | 1 | | | | 1 | | | | | | | | $(3,2).(1^3)$ |
| $(2,1^3).(3)$ | | | | | | | 1 | | | | 1 | 1 | | | | 1 | | | | | | | $(4,1).(1^3)$ |
| $(1^5).(3)$ | | | | | | | | | | | | 1 | | | | | 1 | | | | | | $(5).(1^3)$ |
| $(5).(2,1)$ | | 1 | 1 | 1 | | 1 | | | | | | | | | | | | | | | | | $(1^5).(2,1)$ |
| $(4,1).(2,1)$ | | | 1 | 1 | 1 | 2 | 1 | | 1 | 1 | 1 | | | | | | | | | | | | $(2,1^3).(2,1)$ |
| $(3,2).(2,1)$ | | | | | 1 | 1 | | 1 | 2 | 1 | 1 | | 1 | 1 | 1 | | | | | | | | $(2^2,1).(2,1)$ |
| $(3,1^2).(2,1)$ | | | | | | 1 | 1 | | 1 | 1 | 2 | 1 | 1 | 1 | 1 | 1 | | | | | | | $(3,1^2).(2,1)$ |
| $(4).(4)$ | 1 | 1 | 1 | | 1 | | | 1 | | | | | | | | | | | | | | | $(1^4).(1^4)$ |
| $(3,1).(4)$ | | 1 | 1 | 1 | 1 | 1 | | | 1 | | | | | | | | | | | | | | $(2,1^2).(1^4)$ |
| $(2^2).(4)$ | | | 1 | | | 1 | | | | 1 | | | | | | | | | | | | | $(2^2).(1^4)$ |
| $(2,1^2).(4)$ | | | | 1 | | 1 | 1 | | | | 1 | | | | | | | | | | | | $(3,1).(1^4)$ |
| $(1^4).(4)$ | | | | | | | 1 | | | | | 1 | | | | | | | | | | | $(4).(1^4)$ |
| $(3,1).(3,1)$ | | | 1 | 1 | 1 | 2 | 1 | 1 | 2 | 1 | 1 | | 1 | 1 | | | | | | | | | $(2,1^2).(2,1^2)$ |
| $(2^2).(3,1)$ | | | | | 1 | 1 | | | 1 | 1 | 1 | | 1 | | 1 | | | | | | | | $(2^2).(2,1^2)$ |
| $(2,1^2).(3,1)$ | | | | | | 1 | 1 | | 1 | 1 | 2 | 1 | | 1 | 1 | 1 | | | | | | | $(3,1).(2,1^2)$ |
| $(2^2).(2^2)$ | | | | | | | | 1 | 1 | 1 | | | | 1 | 1 | | | 1 | | | | | $(2^2).(2^2)$ |
| | $(1^8)$ | $(2,1^6)$ | $(2^2,1^4)$ | $(3,1^5)$ | $(2^3,1^2)$ | $(3,2,1^3)$ | $(4,1^4)$ | $(2^4)$ | $(3,2^2,1)$ | $(3^2,1^2)$ | $(4,2,1^2)$ | $(5,1^3)$ | $(3^2,2)$ | $(4,2^2)$ | $(4,3,1)$ | $(5,2,1)$ | $(6,1^2)$ | $(4^2)$ | $(5,3)$ | $(6,2)$ | $(7,1)$ | $(8)$ | |

8. $n+m=9$.

| | $(9)$ | $(8,1)$ | $(7,2)$ | $(7,1^2)$ | $(6,3)$ | $(6,2,1)$ | $(6,1^3)$ | $(5,4)$ | $(5,3,1)$ | $(5,2^2)$ | $(5,2,1^2)$ | $(5,1^4)$ | $(4^2,1)$ | $(4,3,2)$ | $(4,3,1^2)$ | $(4,2^2,1)$ | $(4,2,1^3)$ | $(4,1^5)$ | $(3^3)$ | $(3^2,2,1)$ | $(3^2,1^3)$ | $(3,2^3)$ | $(3,2^2,1^2)$ | $(3,2,1^4)$ | $(3,1^6)$ | $(2^4,1)$ | $(2^3,1^3)$ | $(2^2,1^5)$ | $(2,1^7)$ | $(1^9)$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $(8).(1)$ | 1 | 1 | | | | | | | | | | | | | | | | | | | | | | | | | | | | | $(1)^8.(1)$ |
| $(7,1).(1)$ | | 1 | 1 | 1 | | | | | | | | | | | | | | | | | | | | | | | | | | | $(2,1^6).(1)$ |
| $(6,2).(1)$ | | | 1 | | 1 | 1 | | | | | | | | | | | | | | | | | | | | | | | | | $(2^2,1^4).(1)$ |
| $(6,1^2).(1)$ | | | | 1 | | 1 | 1 | | | | | | | | | | | | | | | | | | | | | | | | $(3,1^5).(1)$ |
| $(5,3).(1)$ | | | | | 1 | | | 1 | 1 | | | | | | | | | | | | | | | | | | | | | | $(2^3,1^2).(1)$ |
| $(5,2,1).(1)$ | | | | | | 1 | | | 1 | 1 | 1 | | | | | | | | | | | | | | | | | | | | $(3,2,1^3).(1)$ |
| $(5,1^3).(1)$ | | | | | | | 1 | | | | 1 | 1 | | | | | | | | | | | | | | | | | | | $(4,1^4).(1)$ |
| $(4^2).(1)$ | | | | | | | | 1 | | | | | 1 | | | | | | | | | | | | | | | | | | $(2^4).(1)$ |
| $(4,3,1).(1)$ | | | | | | | | | 1 | | | | 1 | 1 | 1 | | | | | | | | | | | | | | | | $(3,2^2,1).(1)$ |
| $(4,2^2).(1)$ | | | | | | | | | | 1 | | | | 1 | | 1 | | | | | | | | | | | | | | | $(3^2,1^2).(1)$ |
| $(4,2,1^2).(1)$ | | | | | | | | | | | 1 | | | | 1 | 1 | 1 | | | | | | | | | | | | | | $(4,2,1^2).(1)$ |
| $(3^2,2).(1)$ | | | | | | | | | | | | | | 1 | | | | | 1 | 1 | | | | | | | | | | | $(3^2,2).(1)$ |
| $(7).(2)$ | 1 | 1 | 1 | | | | | | | | | | | | | | | | | | | | | | | | | | | | $(1^7).(1^2)$ |
| $(6,1).(2)$ | | 1 | 1 | 1 | 1 | 1 | | | | | | | | | | | | | | | | | | | | | | | | | $(2,1^5).(1^2)$ |
| $(5,2).(2)$ | | | 1 | | 1 | 1 | | 1 | 1 | 1 | | | | | | | | | | | | | | | | | | | | | $(2^2,1^3).(1^2)$ |
| $(5,1^2).(2)$ | | | | 1 | | 1 | 1 | | 1 | | 1 | | | | | | | | | | | | | | | | | | | | $(3,1^4).(1^2)$ |
| $(4,3).(2)$ | | | | | 1 | | | 1 | 1 | | | | 1 | 1 | | | | | | | | | | | | | | | | | $(2^3,1).(1^2)$ |
| $(4,2,1).(2)$ | | | | | | 1 | | | 1 | 1 | 1 | | 1 | 1 | 1 | 1 | | | | | | | | | | | | | | | $(3,2,1^2).(1^2)$ |
| $(4,1^3).(2)$ | | | | | | | 1 | | | | 1 | 1 | | | 1 | | 1 | | | | | | | | | | | | | | $(4,1^3).(1^2)$ |
| $(3^2,1).(2)$ | | | | | | | | | 1 | | | | | 1 | 1 | | | | 1 | 1 | | | | | | | | | | | $(3,2^2).(1^2)$ |
| $(3,2^2).(2)$ | | | | | | | | | | 1 | | | | 1 | | 1 | | | | 1 | | 1 | | | | | | | | | $(3^2,1).(1^2)$ |
| $(3,2,1^2).(2)$ | | | | | | | | | | | 1 | | | | 1 | 1 | 1 | | | 1 | 1 | | 1 | | | | | | | | $(4,2,1).(1^2)$ |
| $(3,1^4).(2)$ | | | | | | | | | | | | 1 | | | | | 1 | 1 | | | 1 | | | 1 | | | | | | | $(5,1^2).(1^2)$ |
| $(2^3,1).(2)$ | | | | | | | | | | | | | | | | 1 | | | | | | 1 | 1 | | | 1 | | | | | $(4,3).(1^2)$ |
| $(2^2,1^3).(2)$ | | | | | | | | | | | | | | | | | 1 | | | | | | 1 | 1 | | | 1 | | | | $(5,2).(1^2)$ |
| $(2,1^5).(2)$ | | | | | | | | | | | | | | | | | | 1 | | | | | | 1 | 1 | | | 1 | | | $(6,1).(1^2)$ |
| $(1^7).(2)$ | | | | | | | | | | | | | | | | | | | | | | | | | 1 | | | | 1 | | $(7).(1^2)$ |
| $(6).(3)$ | 1 | 1 | 1 | | 1 | | | | | | | | | | | | | | | | | | | | | | | | | | $(1^6).1^3)$ |
| $(5,1).(3)$ | | 1 | 1 | 1 | 1 | 1 | | 1 | 1 | | | | | | | | | | | | | | | | | | | | | | $(2,1^4).(1^3)$ |
| $(4,2).(3)$ | | | 1 | | 1 | 1 | | 1 | 1 | 1 | | | 1 | 1 | | | | | | | | | | | | | | | | | $(2^2,1^2).)1^3)$ |
| $(4,1^2).(3)$ | | | | 1 | | 1 | 1 | | 1 | | 1 | | 1 | | 1 | | | | | | | | | | | | | | | | $(3,1^3).(1^3)$ |
| | $(1^9)$ | $(2,1^7)$ | $(2^2,1^5)$ | $(3,1^6)$ | $(2^3,1^3)$ | $(3,2,1^4)$ | $(4,1^5)$ | $(2^4,1)$ | $(3,2^2,1^2)$ | $(3^2,1^3)$ | $(4,2,1^3)$ | $(5,1^4)$ | $(3,2^3)$ | $(3^2,2,1)$ | $(4,2^2,1)$ | $(4,3,1^2)$ | $(5,2,1^2)$ | $(6,1^3)$ | $(3^3)$ | $(4,3,2)$ | $(5,2^2)$ | $(4^2,1)$ | $(5,3,1)$ | $(6,2,1)$ | $(7,1^2)$ | $(5,4)$ | $(6,3)$ | $(7,2)$ | $(8,1)$ | $(9)$ | |

→

| | $(2^3).(1^3)$ | $(3,2,1).(1^3)$ | $(4,1^2).(1^3)$ | $(3^2).(1^3)$ | $(4,2).(1^3)$ | $(5,1).(1^3)$ | $(6).(1^3)$ | $(1^6).(2,1)$ | $(2,1^4).(2,1)$ | $(2^2,1^2).(2,1)$ | $(3,1^3).(2,1)$ | $(2^3).(2,1)$ | $(3,2,1).(2,1)$ | $(1^5).(1^4)$ | $(2,1^3).(1^4)$ | $(2^2,1).(1^4)$ | $(3,1^2).(1^4)$ | $(3,2).(1^4)$ | $(4,1).(1^4)$ | $(5).(1^4)$ | $(1^5).(2,1^2)$ | $(2,1^3).(2,1^2)$ | $(2^2,1).(2,1^2)$ | $(3,1^2).(2,1^2)$ | $(3,2).(2,1^2)$ | $(4,1).(2,1^2)$ | $(5).(2,1^2)$ | $(1^5).(2^2)$ | $(2,1^3).(2^2)$ | $(2^2,1).(2^2)$ | $(3,1^2).(2^2)$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $(1^9)$ | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | $(6)$ |
| $(2,1^7)$ | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | $(8,1)$ |
| $(2^2,1^5)$ | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | $(7,2)$ |
| $(2^3,1^3)$ | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | $(6,3)$ |
| $(2^4,1)$ | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | $(5,4)$ |
| $(3,1^6)$ | | | | | | | 1 | | | | | | | | | | | | | | | | | | | | 1 | | | | | $(7,1^2)$ |
| $(3,2,1^4)$ | | | | | | 1 | | | | | | | | | | | | | | | | | | | | 1 | 1 | | | | | $(6,2,1)$ |
| $(3,2^2,1^2)$ | | | | | 1 | | | | | | | | 1 | | | | | | | | | | | | 1 | 1 | | | | | 1 | $(5,3,1)$ |
| $(3,2^3)$ | | | | 1 | | | | | | | | | 1 | | | | | | | | | | | | 1 | | | | | 1 | | $(4^2,1)$ |
| $(3^2,1^3)$ | | | 1 | | | | | | | | | | 1 | | | | | | | | | | | 1 | | 1 | | | | | | $(5,2^2)$ |
| $(3^2,2,1)$ | | 1 | | | | | | | | | | 1 | 2 | | | | | | | | | | 1 | 1 | 1 | | | | | 1 | 1 | $(4,3,2)$ |
| $(3^3)$ | 1 | | | | | | | | | | | | 1 | | | | | | | | | | 1 | | | | | | | | 1 | $(3^3)$ |
| $(4,1^5)$ | | | | | | 1 | 1 | | | | | | | | | | | | | 1 | | | | | | 1 | 1 | | | | | $(6,1^3)$ |
| $(4,2,1^3)$ | | | 1 | | 1 | 1 | | | | | 1 | | 1 | | | | | | 1 | | | | | 1 | 1 | 2 | 1 | | | | 1 | $(5,2,1^2)$ |
| $(4,2^2,1)$ | | 1 | | 1 | 1 | | | | | 1 | 1 | | 2 | | | | | 1 | | | | | 1 | 1 | 2 | 1 | | | 1 | 1 | 1 | $(4,3,1^2)$ |
| $(4,3,1^2)$ | | 1 | 1 | | | | | | | 1 | 1 | 1 | 2 | | | | 1 | | | | | 1 | 1 | 2 | 1 | 1 | | | | 1 | 1 | $(4,2^2,1)$ |
| $(4,3,2)$ | 1 | 1 | | | | | | | | 1 | 1 | 1 | 2 | | | 1 | | | | | | 1 | 2 | 1 | 1 | | | | 1 | 1 | 1 | $(3^2,2,1)$ |
| $(4^2,1)$ | | | | | | | | | | 1 | | 1 | 1 | | 1 | | | | | | | 1 | 1 | 1 | | | | | | 1 | | $(3,2^3)$ |
| $(5,1^4)$ | | | 1 | | | 1 | | | | | 1 | | | | | | | | 1 | 1 | | | | 1 | | 1 | | | | | | $(5,1^4)$ |
| $(5,2,1^2)$ | | 1 | 1 | | 1 | | | | 1 | 1 | 2 | | 1 | | | | 1 | 1 | 1 | | | 1 | 1 | 2 | 1 | 1 | | | 1 | | 1 | $(4,2,1^3)$ |
| $(5,2^2)$ | | 1 | | 1 | | | | | 1 | 1 | 1 | | 1 | | | 1 | | 1 | | | | 1 | 1 | 1 | 1 | | | 1 | 1 | 1 | | $(3^2,1^3)$ |
| $(5,3,1)$ | 1 | 1 | | | | | | | 1 | 2 | 1 | 1 | 1 | | 1 | 1 | 1 | | | | 1 | 2 | 2 | 1 | 1 | | | | 1 | 1 | 1 | $(3,2^2,1^2)$ |
| $(5,4)$ | | | | | | | | | | 1 | | 1 | | 1 | 1 | | | | | | | 1 | 1 | | | | | | | 1 | | $(2^4,1)$ |
| $(6,1^3)$ | | | 1 | | | | | | 1 | | 1 | | | | | | 1 | | 1 | | | 1 | | 1 | | | | | | | | $(4,1^5)$ |
| $(6,2,1)$ | | 1 | | | | | | 1 | 2 | 1 | 1 | | | | 1 | 1 | 1 | 1 | | | 1 | 2 | 1 | 1 | | | | 1 | 1 | | | $(3,2,1^4)$ |
| $(6,3)$ | 1 | | | | | | | | 1 | 1 | | | | 1 | 1 | 1 | | | | | 1 | 1 | 1 | | | | | | 1 | | | $(2^3,1^3)$ |
| $(7,1^2)$ | | | | | | | | 1 | 1 | | | | | | 1 | | 1 | | | | 1 | 1 | | | | | | | | | | $(3,1^6)$ |
| $(7,2)$ | | | | | | | | 1 | 1 | | | | | 1 | 1 | 1 | | | | | 1 | 1 | | | | | | 1 | | | | $(2^2,1^5)$ |
| $(8,1)$ | | | | | | | | 1 | | | | | | 1 | 1 | | | | | | 1 | | | | | | | | | | | $(2,1^7)$ |
| $(9)$ | | | | | | | | | | | | | | 1 | | | | | | | | | | | | | | | | | | $(1^9)$ |
| | $(3^2).(3)$ | $(3,2,1).(3)$ | $(3,1^3).(3)$ | $(2^3).(3)$ | $(2^2,1^2).(3)$ | $(2,1^4).(3)$ | $(1^6).(3)$ | $(6).(2,1)$ | $(5,1).(2,1)$ | $(4,2).(2,1)$ | $(4,1^2).(2,1)$ | $(3^2).(2,1)$ | $(3,2,1).(2,1)$ | $(5).(4)$ | $(4,1).(4)$ | $(3,2).(4)$ | $(3,1^2).(4)$ | $(2^2,1).(4)$ | $(2,1^3).(4)$ | $(1^5).(4)$ | $(5).(3,1)$ | $(4,1).(3,1)$ | $(3,2).(3,1)$ | $(3,1^2).(3,1)$ | $(2^2,1).(3,1)$ | $(2,1^3).(3,1)$ | $(1^5).(3,1)$ | $(5).(2^2)$ | $(4,1).(2^2)$ | $(3,2).(2^2)$ | $(3,1^2).(2^2)$ | |

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# THE HEAVISIDE OPERATIONAL CALCULUS.*

By D. G. Bourgin and R. J. Duffin.

In its primary form the Heaviside calculus is concerned with the interpretation and application of functions of the operator $p$ where $p$ takes $x^n$ into $nx^{n-1}$. Various representations are,[1] of course, possible. Heaviside's developments as well as the closely related work of Volterra [2] on permutable functions of the closed cycle, depend on series expansion of $F(p)$ and term by term interpretation according to the association $p^{-\nu} \doteq x^{\nu-1}/\Gamma(\nu)$. The particular representation used in this paper is that of the Laplace-Mellin integrals, namely, $F(p) \doteq f(x)$ [3] stands for

$$F(p) = \int_{-\infty}^{\infty} e^{-xp} f(x)\, dx \tag{1.1}$$

(1.2)[5]

$$\frac{f(x+0) + f(x-0)}{2}{}^{4} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{xp} F(p)\, dp.$$

This paper may be considered a study of some special results in the theory of these integrals.

The specific concerns of this work include the validation of the asymptotic expansion theorem of Heaviside for a wide class of functions and two theorems

* This joint work was completed in all essentials while the junior author, R. J. Duffin, was in the physics department of the University of Illinois. Received by the Editors January 9, 1936; Revised October 5, 1936, and March 6, 1937.

[1] H. Jeffreys, "Operational methods," *Cambridge Tracts*; T. C. Fry, *Annals of Mathematics*, vol. 34 (1921), p. 184; N. Wiener, *Mathematische Annalen*, vol. 95 (1926), p. 95; P. Levy, *Bulletin Mathematique de France*, vol. 1 (1926), p. 174.

[2] Volterra and Peres, *Leçons sur la Composition.*

[3] The notation is due to B. van der Pol, *Philosophical Magazine*, vol. 8 (1929), p. 801.

[4] In this article wherever $f(x)$ stands alone on one side of an equation, the meaning $(f(x+0) + f(x-0))/2$ is to be ascribed to it.

[5] Since there is no finite natural boundary for $F(p)$ in the present work, it is tacitly assumed that analytic continuation is used in the cut plane. For operational application such continuation is usually carried out by the principle of "permanence of form."

Operational interpretations may, of course, be developed for specific function classes by introducing simple closed circuits or Hankel or Pochhammer contours instead of the ordinate $\mathfrak{R}(p) = c$. Such interpretations, in the writers' opinion, are not strictly speaking of "Heaviside" type, in general, since the property of vanishing of the functions, thus defined, for negative real values is given up. Moreover the intimate relationship with the Fourier integral stressed in this paper, is lost.

which may be used to establish many of the formal identities in the literature of the Heaviside theory as well as certain extensions of that discipline for instance to a conjugate Heaviside calculus. The most interesting contributions are those connected with the development of certain reciprocal kernel relationships and the solutions of the Laplace integral equation.

Closely allied to the study of Eqs. 1. 1, 1. 2 is, in a sense, that of Fourier transforms; one writes $e^{-cx}f(x)$ in place of the usual $f(x)$ and $e^{cx}F(p)$ instead of $F(p)$, viz.

$$(1.11) \qquad F(c+it)=\int_{-\infty}^{\infty} f(x)e^{-x(c+it)}dx$$

$$(1.21) \qquad e^{-cx}f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(c+it)e^{itx}dt.$$

These are precisely the integrals of Eqs. 1. 1 and 1. 2.

In general, Fourier integral theorems imply results for the Mellin Transformation. Evidently, then, conditions such as the bounded variation of $f(x)$ in the neighborhood of a point and the absolute integrability of $e^{-cx}f(x)$ over the axis of reals,[6] are sufficient for the inversion formulae, Eq. 1. 11 and Eq. 1. 21, provided that the Cauchy principal value [7] $L_{\lambda\to\infty}\int_{-\lambda}^{\lambda}\int_{-\infty}^{\infty}$ is understood in evaluating the infinite integrals.

In order to extend the set of operators, $f(x)$ in Eqs. 1. 1 and 1. 2 is assumed to vanish on the negative real axis and all "permissible" operators are such as to leave this function class invariant, (i. e. the lower limit of the integral of Eq. 1. 1 may always be taken as 0). This rules out, for instance,

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[6] E. W. Hobson, *Functions of a Real Variable*, Cambridge Press, 2nd edition, vol. 2, p. 721. Throughout this paper the bounded variation condition introduced to guarantee the limit may be generally replaced by any other of the Fourier integral or Fourier Series conditions for convergence at a point.

[7] Some such condition is essential for the integral (with real $f(x)$)

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{ip(x-t)}f(x)\,dx\,dp$$

may be written

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)\cos p(x-t)\,dx\,dp + i\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)\sin p(x-t)\,dx\,dp.$$

The second integral on the right requires much stronger conditions than does the first for convergence. P. Pi Collega, *Mathematische Zeitschrift*, vol. 40 (1935), p. 349. However, the integrand is easily seen to be an odd function in $p$ so that the Cauchy limit on $p$ exists and is 0 for functions satisfying the condition for existence of the Fourier cosine integral.

the operator $e^{hp}$ for $e^{hp}f(x)$ is $f(x+h)$ formally, which is non-zero in general for $(h>0)\ -h<x<0$. However, $e^{-hp}$, $h>0$ is a permissible operator.

We proceed now to establish in a direct fashion, an asymptotic expansion theorem of considerably greater content and precision than Heaviside's "rules."

THEOREM 1. (a) *$F(p)$ is analytic except for poles of order $\mu_1,\cdots,\mu_n$ at $p_1,\cdots,p_n$; $F(p)$ has essential singularities at $e_1,\cdots,e_t$. In each sufficiently small deleted neighborhood $F(p)$ is analytic and expansible in a Laurent series $\sum\limits_0^\infty A_{kn}(p-e_k)^n+\sum\limits_0^\infty B_{kn}(p-e_k)^{-n-1}$. $F(p)$ has branch points of finite order at $b_1,\cdots,b_m$ in the neighborhoods of which*

$$F(p)=(p-b_i)^{-\alpha_i}\psi_i(p-b_i)+\phi_i(p-b_i),\qquad \alpha_i>0;$$

*the power series*

$$\psi_i(p-b_i)=\Sigma\, a_{in}(p-b_i)^n,\qquad \phi_i(p-b_i)=\Sigma\, c_{in}(p-b_i)^n$$

*converge for* $|p-b_i|\leq r_i>0$. (b) $L_{|p|\to\infty}|F(p)|\to 0$ *uniformly for* $\pi/2\leq \arg p-c\leq 3\pi/2$. *If $b$ is the abscissa of the singularity furthest right the ordering is such that $\mathcal{R}(p_j)=\mathcal{R}(b_i)=\mathcal{R}(e_k)=b$ for the first $l$ values of $j$, the first $q$ values of $i$, and the first $f$ values of $k$.*

Under these hypotheses on $F(p)$

$$(2)\quad F(p)\doteq f(x)\sim\sum_{j=1}^{l}x^{\mu_j-1}e^{p_jx}\,\mathrm{Res.}\,F(p)(p-p_j)^{\mu_j-1}/\Gamma(\mu_j)$$
$$+\sum_{i=1}^{q}\sum_{0}^{\infty}e^{b_ix}a_{in}x^{\alpha_i-n-1}/\Gamma(\alpha_i-n)+\sum_{1}^{f}\sum_{0}^{\infty}e^{e_kx}B_{kn}x^n/\Gamma(n+1).$$

For $x>0$ we use the closed contour made up of the ordinate $\mathcal{R}(p)=c>b$, the left hand infinite semi-circle on this ordinate together with the necessary non-intersecting branch cuts and small circles about the poles. By Cauchy's Theorem

$$(2.1)\quad \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{xp}F(p)\,dp-\Sigma\ (\text{resides}+\text{integrals about essential singularities}+\text{integrals around branch cuts}+\text{integrals on the semi-circle})=0.$$

The residues evidently contribute just the terms involved in the first summation in the theorem.

Surround the essential singularities by non-intersecting circles of radii $s_k$ lying within the regions of convergence of the Laurent series. Because of the

uniform convergence of these series on the circles term by term integration is justified. The inverse powers alone contribute to $f(x)$ and their effect is epitomized in the third group of terms of Eq. 2. Evidently

$$L_{n\to\infty} | B_{kn} |^{1/n} = o(s_k).$$

The function $\sum_0^\infty B_{kn}x^n/\Gamma(n+1)$ is then an entire function of minimal type, and thus is dominated, for large $x$ by $Ae^{\epsilon x}$, $\epsilon$ arbitrarily small $> 0$. The minimal property indicates that the contributions of $e_{f+w}$, $(w = 1, \cdots, t - f)$, are to be compared according to the value of the exponentials $e^{-x(b-e_k)}$. For $k = f + w$ these are negligible for large $x$, and accordingly these terms as well as those arising from $p_{l+1}, \cdots, p_n$ are unimportant.

Consider now the third term under the summation sign in Eq. 2. 1. The branch cut for $b_i$ may be taken as a straight line inclined at an angle $\vartheta$ with the real axis $3\pi/2 > \vartheta > \pi/2$. For ease of exposition alone, $\vartheta$ will be taken as $\pi$ in the work below.[8] The contour around the cut may be taken as made up of the part of the upper and lower edges of the cut terminating to the left of $b_i$ within the circle of radius $r_i$ and a loop, denoted hereafter by $C_i$, around the branch point and lying entirely within this circle to complete the cycle. The transformation $z = p - b_i$ brings the branch point $b_i$ to the origin and introduces the factor $e^{b_i x}$. On writing $-\infty \leq \mathcal{R}(z) \leq -\rho_i > -r_i$ with $\arg z = \pi i$ or $-\pi i$ for the upper and lower boundaries of the cut, the absolute value of the contribution due to these parts of the dissected contour may be exhibited as

$$(2.2)\qquad \left| \frac{1}{2\pi i} e^{b_i x} \int_{\rho_i}^{\infty} e^{-rx}[F(re^{i\pi} + b_i) - F(re^{-i\pi} + b_i)]dr \right| \leqq K_1 e^{(b_i-\rho_i)x}/x$$

since $F(p)$ is bounded on the cut away from the branch point by $2\pi K_1$. This is later shown to be negligible in comparison with the other terms in the final developments so that the *behavior of $F(p)$ in the immediate neighborhood* of the branch points determines the result.

For the term $z^{n-a_i}a_{in}$ the loop integral around the branch point may be written, on making the substitution $zx = -u$, as

$$(2.3)\qquad \frac{a_{in}}{2\pi i} e^{b_i x} x^{a_i-n-1} \int_{C'_i} e^{-u}(-u)^{n-a_i} du$$

where $C'_i$ is the Hankel loop starting from the point on the upper edge of the cut (in the $u$ plane) of abscissa $\rho_i x$ and passing counter clockwise about the

[8] Indentations to avoid possible singularities are tacitly neglected since their sole effect is at most a change in $K_1$ of Equation 2. 2.

origin to the point just below on the lower edge. For $x$ large enough this approaches [9]

$$(2.31) \qquad \frac{a_{in}}{\pi} x^{a_i-n-1} \sin \alpha_i\pi \, e^{b_i x} \, \Gamma(n+1-\alpha_i)(-1)^n.$$

For identification as a term in the second expansion in Eq. 2 one need remark here and later that

$$\frac{\sin \alpha_i\pi}{\pi}(-1)^n = \frac{\sin(\alpha_i - n\pi)}{\pi} = [\Gamma(1-(\alpha_i - n))\Gamma(\alpha_i - n)]^{-1}.$$

The maximum difference between Eq. 2.31 and Eq. 2.3 is found, essentially, by bounding $\int_\rho^\infty e^{-rx} r^{n-a_i} dr$. For sufficiently large $x$ this difference is then easily shown to be inferior to

$$(2.32) \qquad K_2 x^\mu e^{(b_i-\rho_i)x}.$$

We wish to show now that the series defined by the sum of terms of Eq. 2.31 exclusive of the $e^{b_i x}$ factor, is an asymptotic series, namely that

$$(2.4) \qquad L_{x\to\infty} x^{-a_i+M+1} \left| \frac{1}{2\pi i}\int_{C_i} e^{(zx)} \psi_i(z) z^{-a_i} dz - \frac{1}{\pi}\sum_0^M a_{im} \sin \alpha_i\pi \Gamma(n+1-\alpha_i) x^{a_i-n-1}(-1)^n \right| \to 0$$

where $\psi_i(z) z^{-a_i}$ is written instead of $F(z+b_i)$ since evidently the loop integral for $\phi_i(z)$ is 0.

The expression under the absolute value signs in Eq. 2.4 is surely inferior to

$$(2.5) \qquad \left| \frac{1}{2\pi i}\int_{C_i} e^{zx} z^{-a_i+M+1} \bar{\psi}_i(z) dz \right| + K_2 x^\mu e^{-\rho_i x} + K_1 e^{(b_i-\rho_i)x}/x$$

where $\bar{\psi}_i(z) z^{M+1}$ is the remainder after subtracting off the first $M+1$ terms of the series expansion of $\psi_i(z)$. Our immediate problem is to show that Eq. 2.5 is at most $o(x^{a_i-M-1})$ for $x \to \infty$. Hence the last terms in that expression is unimportant.

Evidently $\bar{\psi}(z)$ is analytic for $|z| \leq \rho_i$ and so $\bar{\psi}(z) < K_3$. We may deform the loop into a circle $|z| = \sigma_i/x$, $\sigma_i < \rho_i$ and the upper and lower edges of the cut in the range $-\rho_i \leq \Re(z) \leq -\sigma_i/x$.

The contribution from the integrals along the cut is less than

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[9] Whittaker and Watson, *Modern Analysis*, 3rd edition, p. 244.

$$\left| \frac{\sin \alpha_i \pi}{\pi} x^{a_i - M - 2} K_3 \int_{\sigma_i}^{\rho_i x} e^{-r} r^{-a_i + M + 1} dr \right| \tag{2.6}$$
$$\leq K_3 \left| x^{a_i - M - 2} \int_{\sigma_i}^{\infty} e^{-r} r^{-a_i + M + 1} dr \right| \leq K_4 \left| x^{a_i - M - 2} \right|.$$

Similarly the integral around the small circle is inferior to

$$K_3 \sigma^{-a_i + M + 2} x^{a_i - M - 2}. \tag{2.61}$$

The dominants found in Eqs. 2. 6 and 2. 61 guarantee that Eq. 2. 5 is at most $o(x^{a_i - M - 1})$ for $x \to \infty$. The asymptotic character of the expansions considered has thus been established.

According to a simple extension of a classical lemma due to Jordan the integral on the infinite semi-circle vanishes when account is taken of (b). Furthermore, since the dominants given in Eqs. 2. 2 and 2. 22 involve an exponential decrease faster than that of terms with an $e^{b_i x}$ factor, it follows that the value of the Mellin integral in Eq. 2. 1 is asymptotically approximated by the expansions in the statement of the theorem. The assurance that the Mellin integral really exists follows on the observation that the terms under the summation sign in Eq. 2. 1 remain finite for a divergent sequence of sufficiently large semi circles erected on $\mathfrak{R}(p) = c$ and that Eq. 2. 1 is valid for each of the resulting closed contours.

The proof is complete.[10]

If $F(p)$ is restricted to correspond to a real function $f(x)$ (of the real variable $x$) then $a_{in}$ and $B_{kn}$ are real and each exponential of imaginary argument is replaced by a sine or cosine. This follows immediately on remarking that then

$$\mathfrak{R} F(p) = \mathfrak{R} F(\bar{p}); \qquad \mathfrak{I} F(p) = -\mathfrak{I} F(\bar{p})$$

which imply that $b_i$, $e_i$ and $p_i$ occur in conjugate pairs.

Volterra composition, namely

$$F_1(p) F_2(p) \doteq \int_0^x f_1(z) f_2(x - z)\, dz$$

[10] This clears up the doubts arising in the minds of some of the Heaviside followers, for instance those of Carson, concerning the validity and applicability of the asymptotic Heaviside expansions. Cf. Carson, *Electrical Circuits*, p. 84. Carson's difficulties are but imperfectly answered in Levy's paper, *ibid.*, and the main point is not touched. The situation is, of course, summarized in Equation 2. March, *Bulletin of the Mathematical Society*, vol. 33 (1927), p. 311, has utilized a similar method for a rather restricted case. He does not, moreover, give any exact sufficient conditions for validity and omits the demonstration of the asymptotic property of the development.

follows under suitable restrictions from Eqs. 1. 1 and 1. 2 when it is recalled that $f_1(x)$ and $f_2(x)$ vanish for negative values of their arguments. Here we are interested in the (inverse) composition on the $p$ functions, namely

$$(3)\qquad F_3(p) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F_1(p-z)F_2(z)\,dz = \int_0^\infty f_1(x)f_2(x)e^{-px}dx \doteq f_1(x)f_2(x).$$

THEOREM 2. *If $f_1(x)e^{-c_1x}$, $f_2(x)e^{-c_2x}$ belong to $L_2(0,\infty)$,*

$$(\mathrm{b})\qquad \Re(p) = d > c + c_2, \qquad c > c_1 \geqq c_2$$

*then*[11] *Eq. 3 is valid and $F_3(p)$ is analytic for $\Re(p)$ satisfying* (b) *is at worst $o(1)$ for $|q| \to \infty$, $q = \Im(p)$ and the Mellin integral with $F_3(p)$ is summable $C\,1$ to $f_1(x)f_2(x)$*[4] *wherever this has meaning.*

The proof is immediate, for it

$$(3.1)\qquad F_i(\sigma + it) = \text{l.i.m.}\int_0^\infty f_i(x)e^{-x(\sigma+it)}dx \text{ with } \sigma > c_i$$

it is well known that $F_i(\sigma + it)$ is not only analytic in the half plane $\sigma > c_i$ but that

$$(3.2)^{12}\qquad \int_{\sigma-i\infty}^{\sigma+i\infty} |F(z)|^2\,dt < \infty.$$

This guarantees that $F_1(c+it)$ and $F_2(d+iq-(c+it))$ are Fourier transforms of class $L_2$ on the ordinates provided $c > c_1$ $d - c > c_2$. It is easy to see that the analogue of the Parseval identity becomes[13]

$$(3.3)\qquad L_{A\to\infty}\frac{1}{2\pi i}\int_{-A}^{A} F_1(c+it)F_2(d+iq-(c+it))\,dt = L_{A\to\infty}\frac{1}{2\pi i}\int_{c-iA}^{c+iA} F_1(z)F_2(p-z)\,dz = L_{B\to\infty}\int_0^B f_1(x)f_2(x)e^{-xp}dx.$$

The first integrand evidently belongs to $L_1$ by the Schwarz Inequality. The

[11] Somewhat similar theorems, involving different conditions, in connection with Dirichlet Series, occur in the recent literature. Cf. D. V. Widder, *American Journal of Mathematics*, vol. 49 (1927), p. 321, for the case $f_i(x)$ absolutely integrable; V. Bernstein, *Series Dirichlet*, Appendix I for the case $f_i(x)$ analytic in sectors.

[12] Paley and Wiener, "Fourier transforms," *American Colloquium Publications* (hereafter designated P. W.) Theorem 5.

[13] For instance by paralleling the steps in N. Wiener, *Acta Mathematica*, vol. 118 (1930), p. 55.

last integral may be written $\int_0^\infty g(x)e^{-x(h+iq)}dx$, $g(x) \subset L_1(0, \infty)$ where $h = d - c_1 - c_2 > 0$ and $g(x)$ is the product of two functions each of which belongs to $L_2(o, \infty)$

$$(3.4) \qquad g(x) = (e^{-xc_1}f_1(x))(e^{-c_2x}f_2(x)).$$

Accordingly, the integrals in Eq. 3. 3 not only exist but, by Eq. 3. 4, define an analytic function [14] $F_3(p)$ in the half plane determined by (b). In the event that the first integrand of Eq. 3. 3 or $g(x)$ also belongs to $L_2$ then, of course, $F_3(p)$ belongs to $L_2$ as well.

The deduction $L_{|q|\to\infty} | F_3(p) | = o(1)$, $\mathfrak{R}(p) \geq d$ is correct whenever $f_3(x)$ (here $g(x)$) of Eq. 1. 1 belongs to $L_1(0, \infty)$, viz:

$$\left| \int_0^\infty g(x)e^{-x(h+iq)}dx \right| \leq \left| \int_0^A \right| + \left| \int_A^{A^{-1}} g(x)e^{-x(h+iq)}dx \right| + \left| \int_{A^{-1}}^\infty \right| .$$

For $A$ sufficiently small the moduli of the first and last integrals on the right are inferior to $\epsilon$ uniformly in $q$. The second integral goes to 0, by the Riemann-Lebesgue lemma, for $| q | \to \infty$.

The remarks in the introduction make it clear that C 1 summability follows by direct extension from the known Fourier integral result since $g(x) \subset L_1$.

On making use of the relation $e^{-\lambda p}F(p) = f(x - \lambda)$ we may write Eq. 3 in a form convenient for many applications.

$$(3.01) \qquad \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{-xz}F_1(p-z)F_2(z)dz$$
$$= \int_0^\infty e^{-p\lambda}f_1(\lambda)f_2(\lambda - x)d\lambda = \int_x^\infty e^{-p\lambda}f_1(\lambda)f_2(\lambda - x)d\lambda.$$

The analyticity of $F_i(p)$ in the right half plane indicates that the singularities of $F_1(p-z)$ lie to the right of $\mathfrak{R}(z) = c$ and those of $F_2(z)$ to the left. In the special case of polar singularities and $F_2(z)F_1(p-z) = o(z^{-1})$ uniformly in $\arg z - c$ on either the right or left infinite semi-circle constructed on the diameter $\mathfrak{R}(z) = c$ for instance it is possible to close the contour at infinity and to contract in such wise as not to pass any singularities included in the interior. Thus the closed contour contains all the polar singularities of just one of the functions involved.[15]

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[14] S. Bochner, *Vorlesungen über Four. Int.*, Leipzig, p. 145.

[15] This special case comprehends the usual Heaviside rules. $f_1(x) = 1$, $e^{-ax}$, $x^n$ ($n$ integral) leads to the Cauchy formula, to $F_2(p+a)$ and $(-d/dp)^n$ respectively. For non-integral $n$ in the last example Theorem 2 provides the basis for a theory of fractional differentiation and integration of $p$ functions comparable to that known for the $x$ functions. In this connection compare Equation 5. 1.

The following simple identity may be made the basis for many novel developments as well as for a number of formal results already in the Heaviside literature.

(4)[16] $$\int_0^\infty e^{-\rho_1 x} f_1(x) F_2(\rho_2 + x)\,dx = \int_0^\infty e^{-\rho_2 x} f_2(x) F_1(\rho_1 + x)\,dx.$$

It is convenient to use $\phi_i(z) = e^{-\rho_i z} f_i(z)$. Eq. 4 is then obviously valid when

(4.01) $$\int_0^\infty \int_0^\infty \phi_1(x)\phi_2(\lambda) e^{-x\lambda}\,dx\,d\lambda = \int_0^\infty \int_0^\infty \phi_1(x)\phi_2(\lambda) e^{-x\lambda}\,d\lambda\,dx.$$

The indicated inversion is justified for hypotheses such as[17]

THEOREM 3. *$\phi_i(z)$ is integrable $L_1$ over any finite closed range of positive $z$ values not including 0 or $\infty$ and either side of Eq. 4 exists for absolute values of the integrand.*

A simple extension of Titchmarsh's theorem[18] 2.62 suffices.

THEOREM 3A. $\int_0^\infty |\phi_1(x)|\,dx$, $\int_0^\Lambda |\phi_2(\lambda)|\,d\lambda$ $0 < \Lambda < \infty$, $\int_0^\infty \phi_i(\lambda)\,d\lambda$ $(i = 1, 2)$, *exist as Riemann integrals.*[19]

[16] $f_i(x)$ and $\rho_i$ are treated as real for the proofs below. However, the results are valid generally, on splitting up the integrand into the four combinations

$$\mathfrak{R}\,(\mathfrak{J})\,\phi_1(x) \times \mathfrak{R}\,(\mathfrak{J})\,\phi_2(\lambda),$$

if the hypotheses are satisfied for the separate products. The normal case in operational theory involves only $\rho_i$ complex in which case Theorem 3A alone is affected.

[17] The van der Pol result $\int_0^\infty F(p)\,dp = \int_0^\infty f(x)/x\,dx$ is the special case for which one of the functions is 1 and $\rho_1 = \rho_2 = 0$. This identity has been of extraordinary utility in the work of B. van der Pol, *loc. cit.* and later *Philosophical Magazine* papers. The Riemann-Stieltjes equation

$$h(y) = \int_0^\infty F(p) e^{-py}\,dp = \int_0^\infty f(x)/x + y\,dx$$

arises on taking one of the $\phi$'s as $e^{-hy}$.

[18] E. C. Titchmarsh, *Theory of Functions*, Oxford Press. (Hereafter designated T.).

[19] This theorem admits cases such as $\phi_2(x) = e^{x e^{e^x}} \sin e^{e^{e^x}}$. For this function $\int_0^\infty e^{-xp}\phi_2(x)\,dx$ converges conditionally, only, for $p \geq 0$. Eq. 1.2 with $F_2(p)$ requires generally a summability interpretation. For summability C 1 the validation follows from a result of Hardy's, G. H. Hardy, *Messenger of Mathematics*, vol. 47 (1917), p. 178. Since the central inequality, Eq. 4.2, does not require uniform convergence, it is evident that the hypotheses of Theorem 3 and 3A may be further considerably weakened.

The proof is straightforward. We assert that

$$(4.1) \qquad \int_0^X \int_0^\Lambda \phi_1(x)\phi_2(\lambda)e^{-x\lambda}dx\,d\lambda = \int_0^\Lambda \int_0^X \phi_1(x)\phi_2(\lambda)e^{-x\lambda}d\lambda\,dx.$$

Under our hypotheses both iterated integrals exist for absolute values of the integrand. Hence the integrations may be interpreted in the sense of Lebesgue, but then the integration order is immaterial [20] and this result must hold also for the Riemann interpretation.[21]

After a preliminary integration by parts, it is easy to demonstrate the uniform convergence of $\int_0^\infty \phi_2(\lambda)e^{-x\lambda}d\lambda$ for $x \geqq 0$. Accordingly

$$\left| \int_0^X \phi_1(x) \int_\Lambda^\infty \phi_2(\lambda)e^{-x\lambda}d\lambda\,dx \right| \leqq \int_0^X |\phi_1(x)|\,dx \left| \int_\Lambda^\infty \phi_2(\lambda)e^{-x\lambda}d\lambda \right|$$
$$\leqq \epsilon_1 \int_0^\infty |\phi_1(x)|\,dx < \eta_1$$

for (a) $\Lambda$ fixed $\geqq \Lambda_1$ and all $X$, (b) $X$ fixed and all $\Lambda \geqq \Lambda_2$.

The absolute integrability of $\phi_2(\lambda)$ over finite ranges justifies the assertion that for fixed $\Lambda$, $X_1$ exists such that

$$(4.3) \qquad \left| \int_0^\Lambda \phi_2(\lambda)d\lambda \int_X^\infty \phi_1(x)e^{-x\lambda}dx \right| \leqq \epsilon_2 \int_0^\Lambda |\phi_2(\lambda)|\,d\lambda \leqq \eta_2 \text{ for all } X \geqq X_1.$$

Eqs. 4.1, 4.2 and 4.3 are sufficient to establish the validity of the change in integration order [22] involved in Eq. 4. An obvious generalization to cover the case that $\int_l^\Lambda |\phi_2(\lambda)|\,d\lambda$ exists for $l > 0$, $\Lambda < \infty$ only is included by interchanging the rôles of 0 and $\infty$ in the above proof.

THEOREM 3B. $\phi_i(z)$ belongs to $L_k(0, \infty)$, $1 < k \leqq 2$.

The first half of the conditions of Theorem 3 are easily shown to be satisfied for integrability $L_k$ implies integrability $L_1$ over finite ranges. We show now that the last condition of Theorem 3 is also met.

If $\psi_i(z) = \int_0^\infty e^{-xz}|\phi_i(x)|\,dx$ it is known [23] that

$$\int_0^\infty |\psi_i(z)|^{k'}dz < c\left(\int_0^\infty |\phi_i(z)|^k dz\right)^{k'/k}, \qquad c < \infty$$

[20] T., Theorem 12.6.
[21] T., p. 340.
[22] W. H. Young, *Cambridge Philosophical Transactions*, vol. 21 (1910), p. 48.
[23] G. H. Hardy, *Journal of the London Mathematical Society*, vol. 8 (1933), p. 114.

and $1/k + 1/k' = 1$. It may easily be shown that $\psi_i(z)$ is continuous for $z > 0$, hence $\psi_i(z)$ is measurable and therefore $\psi_i(z)$ belongs to $L_{k'}(0, \infty)$.

By Holder's inequality

$$\int_0^\infty |\psi_1(z)\phi_2(z)|\,dz \leq \left(\int_0^\infty |\psi_2(z)|^{k'}dz\right)^{1/k'}\left(\int_0^\infty |\phi_1(z)|^k\right)^{1/k}$$
$$\leq C\left(\int_0^\infty |\phi_2(z)|^k dz\right)^{1/k}\left(\int_0^\infty |\phi_1(z)|^k dz\right)^{1/k} < \infty.$$

Thus the sufficient hypotheses of Theorem 3 are implied by those of Theorem 3B.

As a first direct application of Theorems 2 and 3, we present an independently interesting development of a calculus, conjugate, in a certain sense, to that of Heaviside. Negative integral powers of the variable are not comprehended [24] by our representation (Eq. 1.1 and Eq. 1.2) of the Heaviside calculus. However, an operational theory can be stated for such functions on interchanging the meaning of operator and variable in Eqs. 1.1 and 1.2 so that $q$ and $x$ correspond to the previous $x$, $p$ respectively. The new variable $(x)$ now ranges over the entire complex plane. An operational expression may be interpreted simultaneously according to both the Heaviside and the conjugate symbolic theories on decomposing the operand

$$\psi(p)g(x) = \psi(p)f(x) + \psi(q)F(x)$$

where $g(x) = f(x) + F(x)$ and the notation for the functions indicates the sets to which $f(x)$ and $F(x)$ belong. This decomposition is no longer unique when fractional powers are present for a distinction regarding domain of consideration must be made for $x^{-\nu}$, $0 < \nu < 1$, for instance, accordingly as it is included in the Heaviside set or the set of the conjugate theory functions.

The following algorithms are straightforward consequences of Theorem 3 for functions fulfilling the restrictions stated there.

$$qF(x) = \frac{d}{d(-x)}F(x) \qquad \frac{d}{dq}f(q) \doteq (x)F(x)$$

(a) $$q^{-1}F(x) = \left(\int_0^\infty e^{-x\lambda}f(\lambda)/\lambda d\lambda\right) = \int_0^\infty F(x+\lambda)\,d\lambda = \int_x^\infty F(t)\,dt.$$

(From the viewpoint of Volterra composition there is a formal analogy here to the usual $p^{-1}f(x)$ interpretation

$$\text{i.e. } p^{-1}f(x) = \int_0^\infty f(t)h(x-t)\,dt\left(= \int_0^X f(x)\,dx\right)$$

[24] Formally $p \log p = -x^{-1}$, but this is not in the domain of Equation 1.1.

with $h(y)$ the unit function vanishing on the negative axis of reals. For the special case of the conjugate theory cf. (a) when $x$ is real, the same composition formula may be considered to apply, but $h(y)$ now represents the unit function for negative real values. The composition property with the reflected unit function holds throughout the conjugate theory when $x$ is real. Vide (b) and (c) below.

$$(5.1)\quad \begin{aligned} &\text{(b)}\ \ (q+c)^{-1}F(x)=\int_0^\infty e^{-c\lambda}F(x+\lambda)\,d\lambda=\int_x^\infty e^{-c(t-x)}F(t)\,dt\\ &\text{(c)}\ \ (q+c)^{-n}F(x)=\left(\int_0^\infty e^{-x\lambda}f(\lambda)/(\lambda+c)^n d\lambda\right)\\ &\qquad\qquad\qquad\ =\int_0^\infty e^{-c\lambda}\lambda^{n-1}F(\lambda+x)/\Gamma(n)\,d\lambda\end{aligned}$$

The last result may be obtained operationally (for $n$ a positive integer) by formally differentiating $(q+c)^{-1}F(x)$ with respect to $-c$. With $c=0$ Eq. 5.1 may be interpreted as a fractional integral of a $p$ function.

Theorem III determines the function associated with $f(x^n)$ at least for $n$ integral. For $n=2$ we start with

$$f_1(x)=\exp(-\lambda^2/x)/4(\pi x)^{1/2}\doteq F_1(p)=\exp(-\lambda p^{1/2})/p^{1/2};$$

whence quite directly

$$4\pi^{1/2}f_2(x^2)\doteq\int_0^\infty F_2(z^2)\exp(-p^2)/z^2dz.$$

The application to constant coefficient differential equations is direct and may be briefly summarized. Consider the differential equation with constant coefficients

$$(5.2)\quad \begin{aligned}&\sum_{i=0}^{n}\alpha_i y^{n-i}(x)=L(d/dx)y=L(-q)y=F(x);\\ &\lim_{x\to\infty} y,\cdots,y^{n-1}\to 0,\qquad \mathcal{R}(x)\leq c.\end{aligned}$$

We may write, on the assumption of no positive real zeros,

$$(5.3)\qquad y(x)=\int_0^\infty e^{-x\lambda}f(\lambda)/L(-\lambda)\,d\lambda,\ \text{where}\ f(\lambda)\doteq F(x).$$

In accordance with Theorem 3 (or Eq. 5.1) this may be expressed as

$$(5.4)\qquad y(x)=\int_0^\infty\frac{dA(\lambda)}{d\lambda}F(x+\lambda)\,d\lambda$$

where $A(\lambda)$ is the ordinary "indicial" function on the Heaviside theory corresponding to $L(-\lambda)$. It is well known in fact that

$$A(\lambda) = \frac{1}{L(0)} + \sum e^{\alpha_j \lambda}/\alpha_j \frac{d}{dq} L(-q)|_{\alpha_j}, \qquad L(-\alpha_j) = 0$$

if the zeros are distinct and not positive real.

However, another viewpoint may be used in connection with Eq. 5.3. Define

$$\psi(\alpha_j x) \equiv e^{-\alpha_j x} \int_{-\alpha_j x}^{\infty - \mathcal{J}(\alpha_j x)} e^{-v}/v \, dv, \qquad \mathcal{J}(v) \equiv \mathcal{J}(-\alpha_j x)$$

then for the case $F(x) = x^{-1}$ Eq. 5.3 reduces to

$$y(x) \equiv B(x) = \sum_j \psi(\alpha_j x) / \frac{d}{dq} L(-q)|_{\alpha_j}$$

for distinct not positive real zeros of $L(-p)$.

This is the analogue of the Heaviside-Carson "indicial" function's derivative. For the equation with a general $F(x)$ the inverse composition process of Theorem 2 gives, as an alternative to the formula of Eq. 5.4, the solution

(5.41) $$y(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} B(t) F(x-t) \, dt.$$

The more usual formulation of the Mellin integrals is connected with the operator $s = xd/dx$ and is expressed

(1.1 bis) $$G(s) = \int_0^\infty v^{s-1} g(v) \, dv$$

(1.2 bis) $$2\pi i \, g(v) = \int_{d-i\infty}^{d+i\infty} v^{-s} G(s) \, ds.$$

Combination of the operators $s$ and $p$ may well be expected to have special interest in operational theory. Consider then

(6) $$\gamma(s, p) = \int_0^\infty v^{s-1} e^{-vp} \phi(v) \, dv.$$

Some striking inversion relations arise through the intermediation of Eq. 6. We write

(6.1) $$\psi(s) = \int_{A_1}^{A_2} \gamma(s, \lambda) f(\lambda) \, d\lambda.$$

The nature of the reciprocal formula is indicated by the following purely formal developments

(6.11) $$\psi(s) = \int_{A_1}^{A_2} \int_0^\infty v^{s-1} e^{-v\lambda} \phi(v) f(\lambda) \, dv d\lambda = \int_0^\infty v^{s-1} \phi(v) F(v) \, dv$$

if Theorem 3 applies, where

$$F(v) = \int_{A_1}^{A_2} e^{-xv} f(x)\,dx.$$

Then

$$\phi(v)F(v) = (1/2\pi i)\int_{d-i\infty}^{d+i\infty} v^{-s}\psi(s)\,ds$$

and

$$f(x) = (1/2\pi i)^2 \int_{c-i\infty}^{c+i\infty}\int_{d-i\infty}^{d+i\infty} e^{xv}v^{-s}\psi(s)/\phi(v)\,ds\,dv$$

$$= (1/2\pi i)\int_{d-i\infty}^{d+i\infty} K(s,x)\psi(s)\,ds \tag{6.2}$$

with

$$K(s,x) = (1/2\pi i)\int_{c-i\infty}^{c+i\infty} e^{xv}v^{-s}/\phi(v)\,dv.$$[25]

The rigorous validation of this mode of derivation of the important Eq. 6.2 presents difficulties because the integrals involved are generally not absolutely convergent.[26] For the case $\phi(v) = (1 - e^{-v})^{-1}$,

$$\gamma(s,v) = \Gamma(s)\zeta(s,v) \quad \text{where} \quad \zeta(s,v) = \sum_{n=0}^{\infty}(v+n)^{-s}$$

is the generalized zeta-function, Eq. 6.2 takes the elegant form

$$f(x) = (1/2\pi i)\int_{d-i\infty}^{d+i\infty} [x^{s-1} - \mathbf{1}(x-1)^{s-1}]\psi(s)\,ds \tag{6.21}$$
$$\mathbf{1}(z) = 0 \text{ for } z < 0, = 1 \text{ for } z > 0, \text{ and } = 1/2 \text{ for } z = 0.$$

THEOREM 4. *Eq. 6.21 is valid if*

$$x, x-1, \cdots, x-r \ (x - r \geq A_1 > x - r - 1)$$

*are interior to neighborhoods of bounded variation of $f(x)$ and $A_1 > 0$, $A_2 = \infty$ and furthermore that $f(x)$ belongs to $L_1(A_1, \infty)$, provided that $\mathfrak{R}(s) = d > 1$.*

Consider

$$(1/2\pi i)\int_{d-i\beta}^{d+i\beta} x^{s-1}\psi(s)\,ds = (1/2\pi i)\int_{d-i\beta}^{d+i\beta} x^{s-1}\int_{A_1}^{\infty} f(\lambda)\zeta(s,\lambda)\,d\lambda\,ds \tag{6.3}$$
$$= (1/2\pi i)\int_{A_1}^{\infty} f(\lambda)\int_{d-i\beta}^{d+i\beta} x^{s-1}\zeta(s,\lambda)\,ds\,d\lambda$$
$$= (1/2\pi i)\int_{A_1}^{\infty} f(\lambda)\int_{d-i\beta}^{d+i\beta}\sum_{0}^{\infty}(\lambda+n)^{-s}x^{s-1}\,ds\,d\lambda.$$

The inversion of order of integration is justified by the observation that for

[25] If $\phi(v) = 1/\sum_{0}^{\infty} A_k e^{-B_k v}$ then $K(s,x) = \sum_{0}^{\infty} A_k \cdot (x - B_k)\mathbf{1}(x - B_k)$ formally.

[26] In fact $L_{|t|\to\infty}\,\gamma(d + it, p)$ may not exist. K. Ananda-Rau, *Proceedings of the London Mathematical Society*, vol. 19 (1920), p. 114.

$\mathcal{R}(s) = d > 1$, $|\zeta(s, \lambda)|$ is continuous in $\lambda$ and $s$ when $A_1 \leq \lambda \leq \infty$ and accordingly the integrals are absolutely convergent.

On carrying out the integration there results

$$(6.31) \qquad \int_{A_1}^{\infty} f(\lambda) \sum_0^{\infty} \frac{(x/n + \lambda)^d}{\pi x \log (x/n + \lambda)} \sin \beta \log (x/n + \lambda)\, d\lambda.$$

The term by term integration is correct for $\sum\limits_0^{\infty} x^{-1}(x/n + \lambda)^s$ is uniformly convergent (in $\mathfrak{I}(s)$) for $\mathcal{R}(s) = d > 1$, $\beta < \infty$.

We may write

$$\left| \sum_{N_x}^{\infty} \frac{(x/n + \lambda)^d}{\pi x \log (x/n + \lambda)} \sin (\beta \log (x/n + \lambda)) \right| < \epsilon_1$$

for $N_x$ ($> x$) sufficiently large, uniformly in $\beta$ and $\lambda$. Thus

$$(6.4) \qquad \int_{A_1}^{\infty} |f(\lambda)| \left| \sum_{N_x}^{\infty} \frac{(x/n + \lambda)^d}{\pi x \log (x/n + \lambda)} \sin (\beta \log x/n + \lambda) \right| d\lambda < \epsilon_2.$$

Since also

$$(6.41) \qquad L_{A \to \infty} \sum_0^{N'} \int_A^{\infty} \to 0$$

uniformly in $\beta$ for finite $N'$, $x$ we may invert the operations in Eq. 6. 31 to get

$$\sum_0^{\infty} \int_{A_1}^{\infty} f(\lambda) \frac{(x/n + \lambda)^d}{\pi x \log (x/n + \lambda)} \sin (\beta \log x/n + \lambda)\, d\lambda$$

or what is essentially the same thing according to Eq. 6. 4

$$(6.5) \qquad \sum_0^{N_x} \int_{A_1}^{\infty} f(\lambda) \frac{(x/n + \lambda)^d}{\pi x \log (x/n + \lambda)} \sin (\beta \log x/n + \lambda)\, d\lambda.$$

Accordingly,

$$\begin{aligned}
(6.32) \quad & L_{\beta \to \infty} (1/2\pi i) \int_{d-i\beta}^{d+i\beta} x^{s-1} \psi(s)\, ds \\
&= L_{\beta \to \infty} \sum_0^{N_x} \int_{A_1}^{\infty} f(\lambda) \frac{(x/n + \lambda)^d}{\pi x \log (x/n + \lambda)} \sin (\beta \log x/n + \lambda)\, d\lambda \\
&= L_{\beta \to \infty} \sum_0^{N_x} \int_{\log x/n + A_1}^{-\infty} e^{(d-1)z} f(xe^{-z} - n) \frac{\sin \beta z}{\pi z}\, dz
\end{aligned}$$

$$\begin{aligned}
(6.6) \quad & \int_{\log x/n + A_1}^{-\infty} | e^{(d-1)z} f(xe^{-z} - n) |\, dz \\
&= \int_{A_1}^{\infty} | x^{d-1} f(\lambda)/(n + \lambda)^d |\, d\lambda < \infty.
\end{aligned}$$

Because of Eq. 6. 6 and Eq. 6. 41 the Riemann-Lebesgue lemma may be applied to show that the Dirichlet integrals in Eq. 6. 32 vanish unless $x \geq n + A_1$. Accordingly, the limit of the first integral in Eq. 6. 32 is

$$(6.7) \qquad f(x) + f(x-1) + \cdots f(x-r)$$

where $x - r \geq A_1 > x - r - 1$, provided the function is of bounded variation in the neighborhoods of the arguments in question.

Similarly

$$(6.8) \quad L_{\beta\to\infty} (1/2\pi i) \int_{d-i\beta}^{d+i\beta} \mathbf{1}(x-1)(x-1)^{s-1}\psi(s)\,ds = f(x-1) + \cdots f(x-r)$$

Subtraction of the expressions in Eq. 6.8 from those in Eq. 6.7 yields the desired result.

THEOREM 4A. *If* $f(x) \subset L_1(A_1, \infty)$, *Eq.* 6.21 *is valid in the sense that the right-hand integral is summable* (C, 1) *wherever* $f(x+0) + f(x-0)$ *exists.*

We have merely to investigate

$$L_{\beta\to\infty} (1/2\pi i) \int_{d-i\beta}^{d+i\beta} \left(1 - \frac{|\mathfrak{I}(s)|}{\beta}\right)(x^{s-1} - \mathbf{1}(x-1)(x-1)^{s-1})\psi(s)\,ds.$$

The steps and reasoning of the argument are precisely the same in detail as in the proof above. The only change is that the resulting integrals (Eq. 6.32 for instance) are of Fejer instead of Dirichlet type.

One immediate application is furnished by the Laplace integral equation, Eq. 1.1 where $p$ is now a real variable, $F(p)$ is supposed known for $p \geq p_0$ and $f(x)$ is required. There is no fundamental restriction in assuming $f(x)$ bounded and of class $L_1(0, \infty)$.[27]

For $f(x)$ satisfying the conditions of Theorems 4 or 4A we may exhibit solutions in the form [28]

$$(7) \quad f(x) = (1/2\pi i) \int_{d-i\infty}^{d+i\infty} (x^{s-1} - \mathbf{1}(x-1)(x-1)^{s-1}) \int_0^\infty (p^{s-1}F(p)/\Gamma(s)(1-e^{-p}))\,dp\,ds$$

or

$$= L_{\beta\to\infty} (1/2\pi i) \int_{d+i\beta}^{d-i\beta} \left(1 - \frac{|\mathfrak{I}(s)|}{\beta}\right)(x^{s-1} - \mathbf{1}(x-1)(x-1)^{s-1}) \int_0^\infty (p^{s-1}F(p)/\Gamma(s)(1-e^{-p}))\,dp\,ds,$$

$$f(x) = 0 \text{ for } x < A_1.$$

These solutions are easily established on noting that for $\phi(v) = (1-e^{-v})^{-1}$

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[27] P. W., p. 37.

[28] Evidently the general formal solution may be written

$$\frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} K(s,x) \int_0^\infty v^{s-1}\phi(v)F(v)\,dv.$$

Theorem 3 applies to the inversion indicated in Eq. 6. 11. In fact it is manifest $|F(v)| < Me^{-A_1v}/v$ and accordingly $\int_0^\infty |v^{s-1}F(v)(1-e^{-v})|\,dv$ is certainly absolutely convergent for $\Re(s) > 2$.

A solution, Eq. 7. 4, formally somewhat similar to that given by Paley and Wiener [29] (who, however, work in the domain of $L_2$ functions) follows on using the specialization $p=1$, $\phi(v)=v$ or $\gamma(s,p)=\Gamma(s+1)$ in Eq. 6. This may be rigorously established for $f(x)$ belonging to $L_1(0,\infty)$ as follows:

$$\int_0^\infty f(x)/(1+x)^{s+1}dx = [1/\Gamma(s+1)]\int_0^\infty e^{-\lambda}\lambda^s F(\lambda)\,d\lambda = \psi(s) \tag{7.1}$$

by Theorem 3 and Eq. 5. 1 for $\Re(s) > 0$. Writing $x+1=e^z$

$$\psi(s) = \int_0^\infty e^{-sz}f(e^z-1)\,dz. \tag{7.2}$$

Clearly,

$$\int_0^\infty |f(e^z-1)|\,dz = \int_0^\infty |f(x)/1+x| < \int_0^\infty |f(x)|\,dx < \infty.$$

Hence Eq. 1. 2 applies and

$$f(e^z-1) = (1/2\pi i)\int_{c-i\infty}^{c+i\infty} e^{zs}\psi(s)\,ds \qquad c>0, \tag{7.3}$$

or

$$\begin{aligned} f(x) &= (1/2\pi i)\int_{c-i\infty}^{c+i\infty}(x+1)^s\psi(s)\,ds \\ &= (1/2\pi i)\int_{c-i\infty}^{c+i\infty}\frac{(x+1)^s}{\Gamma(s+1)}\int_0^\infty e^{-\lambda}\lambda^s F(\lambda)\,d\lambda\,ds. \end{aligned} \tag{7.4}$$

The result holds when $x$ is interior to a neighborhood of bounded variation for $f(x)$.

Here also a generalization is afforded by replacing convergence by summability $(C,1)$ or Sommerfeld type, in that the last integral of Eq. 7. 4 is summable to $f(x)$ when $f(x+0)+f(x-0)$ has meaning. This observation hinges essentially on the fact that the summability property of Fourier integrals since $f \subset L_1$ is patently directly extensible to Eq. 1. 21 and hence to Eq. 7. 3 and thus to Eq. 7. 4.

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[29] P. W., p. 37-39. (Here references to D. V. Widder's work may be found as well.) The P. W. solution apart from its implication of $L_2$ function classes and an apparent integration order change, is essentially transformable to the type of Eq. 7. 4 with the specialization.

# NOTE ON FORMAL LOGIC.*

By M. H. Stone.

It has been observed that the theory of Boolean algebras assumes a particularly satisfactory algebraic form when developed in terms of the symmetric difference $a + b$ and the product $a \cdot b$ as fundamental operations: for Boolean algebras are then characterized as rings with unit in which every element is idempotent.[1] The close connection between Boolean algebras and the formal (Aristotelian) logic of propositions therefore suggests that a logistic system built up from corresponding operations would be of some interest, and would have the special advantage of reducing the proofs of most logical theorems to simple and essentially familiar algebraic calculations. In the present note we shall develop such a system, based on results of Leśniewski and Bernstein.[2]

Propositions are to be regarded as abstract entities and denoted by the letters $a, b, c, \cdots$. We postulate three primitive operations on propositions, each of which results in a new proposition; and indicate the propositions resulting from their application by $a + b$, $a \cdot b$, and $a'$ respectively. We may read $a + b$ as "$a$ if and only if $b$" or "$a$ is equivalent to $b$"; we may read $a \cdot b$ as "$a$ or $b$"; and we may read $a'$ as "not $a$." To indicate that a particular proposition $a$ is to be placed on the list of asserted propositions we write $\vdash a$. As primitive assertions, we postulate that for arbitrary propositions $a, b, c$ (whether given directly or expressed as "polynomials" in terms of the postulated operations and other directly given propositions)

(1.1) $\vdash [(a + b) + (c + a)] + [b + c]$

(1.2) $\vdash [a + (b + c)] + [(a + b) + c]$

(2.1) $\vdash [a \cdot (b \cdot c)] + [(a \cdot b) \cdot c]$

(2.2) $\vdash [(a + b) \cdot c] + [(c \cdot a) + (c \cdot b)]$

(2.3) $\vdash (a \cdot a) + a$

(3.1) $\vdash [(a + a') \cdot b] + b.$

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[1] Stone, *Transactions of the American Mathematical Society*, vol. 40 (1936), pp. 37-111, especially pp. 39-48.

[2] Leśniewski, *Fundamenta Mathematicae*, vol. 14 (1929), pp. 1-81; B. A. Bernstein, *Annals of Mathematics* (2), vol. 37 (1936), pp. 317-325.

In order to bring other propositions upon the list of asserted propositions, we postulate the informal deductive rules

(A) if $\vdash a$ and $\vdash a + b$, then $\vdash b$;

(B) if $\vdash a$, then $\vdash a \cdot b$.

The application of these rules will be indicated by the schemes

$$\text{(A)}\ \frac{\begin{array}{l}\vdash a\\ \vdash a + b\end{array}}{\vdash b} \qquad\qquad \text{(B)}\ \frac{\vdash a}{\vdash a \cdot b}.$$

We introduce a relation $=$ between propositions as follows:

DEFINITION 1. $a = b$ *if* $\vdash a + b$.

We can then introduce two further operations either through the definitions [3]

DEFINITION 2. $\vdash [a \,\&\, b] + [(a + b) + (a \cdot b)]$,

DEFINITION 3. $\vdash [a \rightarrow b] + [b + (a \cdot b)]$,

or through the equivalent definitions

DEFINITION 2′. $a \,\&\, b = (a + b) + (a \cdot b)$,

DEFINITION 3′. $a \rightarrow b = b + (a \cdot b)$.

The proposition $a \,\&\, b$ may be read "$a$ and $b$," the proposition $a \rightarrow b$ may be read "$a$ implies $b$." We shall see that the interpretations of the primitive and defined operations are all justified by subsequent results.

We commence our investigation by considering the consequences of (1. 1), (1. 2), (A) and Definition 1. The system so described is due to Leśniewski.[4] We obtain the following fundamental result:

THEOREM 1. *In terms of the operation* $+$ *and the relation* $=$, *taken as an equality-relation, the system under consideration is an additive abelian group in which every element is of order 2. If the zero element of this group*

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[3] The usual form of definition would be to describe $a \,\&\, b$, $a \rightarrow b$ as abbreviations for $(a + b) + (a \cdot b)$, $b + (a \cdot b)$ respectively. For comments on the present form, which is better suited to our later algebraic considerations, if not to the requirements of a strictly formal logic, see Tarski (Tajtelbaum), *Fundamenta Mathematicae*, vol. 4 (1923), pp. 196-200, especially p. 197.

[4] Leśniewski, *loc. cit.* We write $+$ in place of his $\equiv$.

*be denoted by* 0, *the statements* $a = 0$ *and* $\vdash a$ *are equivalent. In particular, we have, for all* $a, b, c, d$,

($\alpha$) $a = a$; ($\alpha'$) *if* $a = b$, *then* $b = a$;
($\alpha''$) *if* $a = b$ *and* $b = c$, *then* $a = c$;
($\beta$) *if* $a = c$ *and* $b = d$, *then* $a + b = c + d$;
($\gamma$) $a + b = b + a$; ($\delta$) $a + (b + c) = (a + b) + c$;
($\epsilon$) *the equation* $x + b = a$ *has* $a + b$ *as a solution.*

It is well known that the properties ($\alpha$)–($\delta$), together with the existence of a solution of the equation $x + b = a$, are characteristic for abelian groups.[5] They imply that the solution of $x + b = a$ is unique (in the sense that $x + b = a$ and $y + b = a$ imply $x = y$) and that the zero element 0 exists and satisfies the equation $x + a = a$ for every $a$. Thus the property ($\epsilon$) above yields the special relation $a + a = 0$ for every $a$; in other words, every element is of order 2. Accordingly, we need establish only the properties ($\alpha$)–($\epsilon$).

We begin with several lemmas, as follows:

(1.3) $\vdash (a + b) + (b + a)$;
(A′) *if* $\vdash a + b$, *then* $\vdash b + a$;
(A″) *if* $\vdash b$ *and* $\vdash a + b$, *then* $\vdash a$;
(1.4) $\vdash a + a$;
(1.5) $\vdash [b + a] + [(c + b) + (a + c)]$;
(1.6) $\vdash a + [b + (b + a)]$.

Ad (1.3). Substituting $b, a, b$ for $a, b, c$ respectively in (1.2), and $b, a + b, b + a$ for $a, b, c$ respectively in (1.1), we obtain the scheme

$$(\mathrm{A})\ \frac{\begin{array}{l}\vdash [b + (a + b)] + [(b + a) + b] \\ \vdash \{[b + (a + b)] + [(b + a) + b]\} + \{(a + b) + (b + a)\}\end{array}}{\vdash (a + b) + (b + a).}$$

Ad (A′). Using (1.3) we have the scheme

$$(\mathrm{A})\ \frac{\begin{array}{l}\vdash a + b \\ \vdash (a + b) + (b + a)\end{array}}{\vdash b + a.}$$

Ad (A″). We now have the scheme

$$(\mathrm{A})\ \frac{\vdash b \quad (\mathrm{A}')\,\dfrac{\vdash a + b}{\vdash b + a}}{\vdash a.}$$

[5] See, for instance, van der Waerden, *Moderne Algebra*, Berlin, 1930, vol. I, pp. 15-19.

Ad (1.4). Substituting $b$, $a$ for $a$, $b$ respectively in (1.3) and $b$, $a$, $a$ for $a$, $b$, $c$ respectively in (1.1), we obtain the scheme

$$\text{(A)}\ \frac{\begin{array}{l}\vdash (b+a)+(a+b)\\ \vdash [(b+a)+(a+b)]+[a+a]\end{array}}{\vdash a+a.}$$

Ad (1.5). Substituting $c$, $b$, $a$ for $a$, $b$, $c$ respectively in (1.1), we obtain the scheme

$$\text{(A}'\text{)}\ \frac{\vdash [(c+b)+(a+c)]+[b+a]}{\vdash [b+a]+[(c+b)+(a+c)]}.$$

Ad (1.6). Using (1.3) and substituting $a$, $b$, $b+a$ for $a$, $b$, $c$ respectively in (1.2), we obtain the scheme

$$\text{(A}''\text{)}\ \frac{\begin{array}{r}\vdash (a+b)+(b+a)\\ \vdash \{a+[b+(b+a)]\}+\{(a+b)+(b+a)\}\end{array}}{\vdash a+[b+(b+a)].}$$

This completes the proof of the lemmas listed above.

We turn now to the properties ($\alpha$)–($\epsilon$), taking them up in a somewhat altered order.

Ad ($\alpha$). By (1.4) and Def. 1, we have $a = a$.

Ad ($\alpha'$). By Def. 1, $a = b$ means $\vdash a + b$. Hence $a = b$ implies $\vdash b + a$ by (A$'$) and thus $b = a$ by Def. 1.

Ad ($\alpha''$). If $a = b$ and $b = c$, then $\vdash b + a$ and $\vdash c + b$ by ($\alpha'$) and Def. 1. Hence, by using (1.5), we obtain the scheme

$$\text{(A)}\ \frac{\begin{array}{l}\vdash b+a\\ \vdash [b+a]+[(c+b)+(a+c)]\end{array}}{\vdash (c+b)+(a+c)}$$

$$\text{(A)}\ \frac{\vdash c+b}{\vdash a+c.}$$

Thus $a = c$ by Def. 1.

Ad ($\gamma$). By (1.3) and Def. 1, we have $a + b = b + a$.

Ad ($\delta$). By (1.2) and Def. 1, we have $a + (b + c) = (a + b) + c$.

Ad ($\beta$). If $a = c$, we have $\vdash a + c$ by Def. 1. On substituting $c$, $a$, $b$ for $a$, $b$, $c$ respectively in (1.5), we obtain the scheme

$$\text{(A)}\ \frac{\begin{array}{l}\vdash a+c\\ \vdash [a+c]+[(b+a)+(c+b)]\end{array}}{\vdash (b+a)+(c+b).}$$

By Def. 1, we have $b + a = c + b$; and by $(\gamma)$ and $(\alpha'')$ we infer that $a + b = c + b$. If $b = d$, we can substitute $b, c, d$ for $a, b, c$, respectively in this equation, obtaining $b + c = d + c$. Applying $(\gamma)$ and $(\alpha'')$, we infer that $c + b = c + d$. Thus a final application of $(\alpha'')$ yields $a + b = c + d$.

Ad $(\epsilon)$. By $(\alpha)$, $(\beta)$, and $(\gamma)$, we have $(a + b) + b = (b + a) + b$. By $(\alpha'')$ and $(\gamma)$, we then have $(a + b) + b = b + (b + a)$. On the other hand, (1.6) and Def. 1 yield $a = b + (b + a)$. Hence $(\alpha')$ and $(\alpha'')$ yield $(a + b) + b = a$.

We still have to establish the equivalence of $a = 0$ and $\vdash a$. From the preceding results, we know that $a = 0$ if and only if $a = a + a$; and that the statements $a = a + a$ and $\vdash a + (a + a)$ are equivalent. Now, using (1.4) and substituting $a, a$ for $a, b$ respectively in (1.6), we obtain the schemes

$$(\mathrm{A}'')\ \frac{\begin{array}{c}\vdash a + a\\ \vdash a + (a + a)\end{array}}{\vdash a} \qquad\qquad (\mathrm{A})\ \frac{\begin{array}{c}\vdash a\\ \vdash a + [a + (a + a)]\end{array}}{\vdash a + (a + a)}.$$

Hence the statements $\vdash a$ and $\vdash a + (a + a)$ are equivalent. It follows that the statements $a = 0$ and $\vdash a$ are equivalent.

We next consider the effect of introducing (2.1), (2.2), (2.3) and (B) into the system studied in Theorem 1. We obtain the following fundamental result:

THEOREM 2. *In terms of the operations* $+$ *and* $\cdot$ *and of the relation* $=$ *of Definition* 1, *the system under consideration is a Boolean ring—that is, a ring (necessarily commutative) in which every element is idempotent.*[6] *In particular, we have, for all* $a, b, c$,

$(\eta)$ *if* $a = c$ *and* $b = d$, *then* $a \cdot b = c \cdot d$;
$(\zeta)$ $a \cdot b = b \cdot a$; $\quad(\iota)$ $a \cdot (b \cdot c) = (a \cdot b) \cdot c$;
$(\kappa)$ $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$;
$(\lambda)$ $a \cdot a = a$.

It is well known that the properties $(\alpha)$–$(\delta)$, $(\eta)$–$(\kappa)$, together with the existence of a solution of the equation $x + b = a$ are characteristic for a commutative ring.[7] We may remark that $(\lambda)$ implies $a + a = 0$ and hence $(\epsilon)$: for obvious applications of $(\alpha'')$, $(\beta)$, $(\zeta)$, $(\kappa)$, and $(\lambda)$ yield

[6] Stone, *loc. cit.*
[7] See, for instance, van der Waerden, *Moderne Algebra*, Berlin, 1930, vol. I, pp. 36-40.

$$\begin{aligned} a+a &= (a+a)\cdot(a+a) = [(a+a)\cdot a]+[(a+a)\cdot a] \\ &= [a\cdot(a+a)]+[a\cdot(a+a)] = [(a\cdot a)+(a\cdot a)]+[(a\cdot a)+(a\cdot a)] \\ &= [a+a]+[a+a]. \end{aligned}$$

We now discuss the indicated properties in a somewhat altered order.

Ad ($\iota$). By (2.1) and Def. 1, we have $a\cdot(b\cdot c)=(a\cdot b)\cdot c$.

Ad ($\lambda$). By (2.3) and Def. 1, we have $a\cdot a=a$.

Ad ($\zeta$). By ($\lambda$) we have $(a+b)\cdot(a+b)=a+b$. By (2.2) and Def. 1, we have $(a+b)\cdot(a+b)=[(a+b)\cdot a]+[(a+b)\cdot b]$, $(a+b)\cdot a=(a\cdot a)+(a\cdot b)$, $(a+b)\cdot b=(b\cdot a)+(b\cdot b)$. Applying ($\alpha'$), ($\alpha''$), ($\beta$), ($\gamma$), ($\delta$), and ($\lambda$) in an obvious manner, we therefore obtain $[(a\cdot b)+(b\cdot a)]+[a+b]=a+b$. Theorem 1 now shows that $(a\cdot b)+(b\cdot a)=0$ and hence that $a\cdot b=b\cdot a$.

Ad ($\kappa$). Substituting $b, c, a$ for $a, b, c$ respectively in (2.2) and applying Def. 1, we have $(b+c)\cdot a=(a\cdot b)+(a\cdot c)$. Then by ($\zeta$) and ($\alpha''$), we have $a\cdot(b+c)=(a\cdot b)+(a\cdot c)$.

Ad ($\eta$). If $a=c$, then $\vdash a+c$ by Def. 1. On substituting $a, c, b$ for $a, b, c$ respectively in (2.2), we therefore have the scheme

$$(\mathrm{B})\ \frac{\vdash\ a+c}{\vdash\ (a+c)\cdot b}$$

$$(\mathrm{A})\ \frac{\vdash[(a+c)\cdot b]+[(b\cdot a)+(b\cdot c)]}{\vdash(b\cdot a)+(b\cdot c).}$$

By Def. 1, we then have $b\cdot a=b\cdot c$. If $b=d$, we can substitute $b, c, d$ for $a, b, c$ respectively in this equation, obtaining $c\cdot b=c\cdot d$. Applying ($\alpha''$) and ($\zeta$) in an obvious way, we obtain $a\cdot b=c\cdot d$.

We observe that the informal rule (B) has been used only in the proof of ($\eta$). It is therefore of particular interest to note further that (B) can be deduced from (1.1), (1.2), (A), (2.2), and ($\eta$), as we shall now show. If $\vdash a$, then $a=0$ by Theorem 1; and $a=a+a$, also by Theorem 1. By ($\alpha$), ($\eta$) and Def. 1, we therefore have $\vdash[a\cdot b]+[(a+a)\cdot b]$. Hence, on substituting $b\cdot a$ for $a$ in (1.4) and $a, a, b$ for $a, b, c$ respectively in (2.2), we obtain the scheme

$$\vdash(b\cdot a)+(b\cdot a)$$

$$(\mathrm{A}'')\ \frac{\vdash[(a+a)\cdot b]+[(b\cdot a)+(b\cdot a)]}{\vdash\ (a+a)\cdot b}$$

$$(\mathrm{A}'')\ \frac{\vdash[a\cdot b]+[(a+a)\cdot b]}{\vdash\ a\cdot b.}$$

Thus ($\eta$) and (B) may be regarded as equivalent with respect to the primitive propositions and the informal rule (A) of the system.

We now introduce the last of our primitive propositions, (3.1). This proposition is due essentially to Bernstein.[8] We then have

THEOREM 3. *The postulation of* (3.1) *is equivalent to the postulation of a unit $e$ in the Boolean ring of Theorem 2 together with the definition $a' = a + e$.*

By (3.1) and Definition 1, we have $(a + a') \cdot b = b$ for every element $b$. The element $a + a'$ thus has the properties of a unit in the Boolean ring of Theorem 2. Since two units in a commutative ring are necessarily equal,[9] we see that the unit $a + a'$ is independent of $a$. Denoting the unit by $e$, we therefore have $a + a' = e$; and we conclude by Theorem 1 that $a' = a + e$ for every $a$. On the other hand, if the Boolean ring of Theorem 2 has a unit $e$ and $a' = a + e$, we can apply ($\alpha$), ($\alpha'$), ($\alpha''$), ($\beta$), ($\delta$), ($\epsilon$), and ($\eta$) to obtain

$$\begin{aligned}(a + a') \cdot b &= [a + (a + e)] \cdot b = [(a + a) + e] \cdot b = [0 + e] \cdot b \\ &= e \cdot b = b;\end{aligned}$$

and Def. 1 then yields $(3.1) \vdash [(a + a') \cdot b] + b$.

The results obtained in Theorems 1, 2, and 3 may now be inverted as follows:

THEOREM 4. *In an additive abelian group, with* 0 *as its zero element, let the truth of the equation $a = 0$ be indicated by $\vdash a$. If this group has the property that every element is of order* 2, *then* (1.1), (1.2), *and* (A) *are theorems; if this group is a Boolean ring under a suitable multiplication, then* (1.1), (1.2), (2.1), (2.2), (2.3), (A), *and* (B) *are theorems; and, if this group is a Boolean ring with unit under a suitable multiplication, then* (1.1), (1.2), (2.1), (2.2), (2.3), (3.1), (A), *and* (B) *are theorems. In each of these cases, $a = b$ if and only if $a + b = 0$ or $\vdash a + b$.*

The proof may be left to the reader.

In order to illustrate the demonstration of logical theorems, in accordance with the principle established in Theorem 1 that the statements $\vdash a$ and $a = 0$ are equivalent, we give the following result:

THEOREM 5. *In the logistic system under consideration we have for all $a$, $b$, $c$*

[8] B. A. Bernstein, *loc. cit.*

[9] See, for instance, van der Waerden, *Moderne Algebra*, vol. I (Berlin, 1930), p. 40.

(4.1) $\vdash (a' \to a) \to a;$

(4.2) $\vdash a \to (a' \to b);$

(4.3) $\vdash [a \to b] \to [(b \to c) \to (a \to c)];$—

*together with the informal deductive rule*

(C) *if* $\vdash a$ *and* $\vdash a \to b$, *then* $\vdash b$.

*Corresponding to the informal rule* (C) *we have for all* $a$, $b$

(5.1) $\vdash [a \,\&\, (a \to b)] \to b.$

According to Definitions 2, 3 (2′, 3′) and Theorems 1, 2, 3 we have to establish the algebraic identities

(4.1*) $a + [a + (a + e) \cdot a] \cdot a = 0,$

(4.2*) $\big(b + [(a + e) \cdot b]\big) + \big(a \cdot \{b + [(a + e) \cdot b]\}\big) = 0,$

(4.3*) $\big[\{c + (a \cdot c)\} + \{[c + (b \cdot c)] \cdot [c + (a \cdot c)]\}\big]$
$+ \big[\big(b + (a \cdot b)\big) \cdot \big(\{c + (a \cdot c)\} + \{[c + (b \cdot c)] \cdot [c + (a \cdot c)]\}\big)\big] = 0,$

(5.1*) $b + \big[\big(\{a + [b + (a \cdot b)]\} + \{a \cdot [b + (a \cdot b)]\}\big) \cdot b\big] = 0,$

together with the rule

(C*) if $a = 0$ and $b + (a \cdot b) = 0$, then $b = 0$.

Since the operations are to be carried out in a commutative ring with unit $e$, we can expand these expressions in a familiar way—we can drop all brackets (using the convention that multiplications take precedence over additions), we can write the factors of any product in alphabetical order, and we can drop $e$ as a factor from any product in which it occurs. Our alleged identities then assume the respectively equivalent forms

(4.1**) $a + a \cdot a + a \cdot a \cdot a + a \cdot a = 0,$

(4.2**) $b + a \cdot b + b + a \cdot b + a \cdot a \cdot b + a \cdot b = 0,$

(4.3**) $c + a \cdot c + c \cdot c + a \cdot c \cdot c + b \cdot c \cdot c + a \cdot b \cdot c \cdot c + b \cdot c$
$+ a \cdot b \cdot c + b \cdot c \cdot c + a \cdot b \cdot c \cdot c + b \cdot b \cdot c \cdot c$
$+ a \cdot b \cdot b \cdot c \cdot c + a \cdot b \cdot c + a \cdot a \cdot b \cdot c + a \cdot b \cdot c \cdot c$
$+ a \cdot a \cdot b \cdot c \cdot c + a \cdot b \cdot b \cdot c \cdot c + a \cdot a \cdot b \cdot b \cdot c \cdot c = 0,$

(5.1**) $b + a \cdot b + b \cdot b + a \cdot b \cdot b + a \cdot b \cdot b + a \cdot a \cdot b \cdot b \cdot = 0.$

By application of the special rules $a \cdot a = a$, $a + a = 0$, these relations are seen to be identities, as we wished to prove. As to (C*), we note that $a = 0$ implies $b + (a \cdot b) = b + (0 \cdot b) = b + 0 = b$, and hence that $a = 0$ and $b + (a \cdot b) = 0$ together imply $b = 0$.

It has been shown by Łukasiewicz and Tarski [10] that a complete logistic system for the Aristotelian logic of propositions can be based on the primitive operations $\rightarrow$ and $'$ with (4.1), (4.2), (4.3) as primitive propositions and (C) as the sole informal deductive rule. Hence the system discussed here contains all of the ordinary logic of propositions. Our system has a similar relation to that of Russell and Whitehead, whose primitive operations correspond to our $\cdot$ and $'$. On the other hand, it can be shown that the Łukasiewicz-Tarski and Russell-Whitehead systems contain ours (under suitable definitions of $+$ and $\cdot$ when not taken as primitive). Since this aspect of the situation is quite familiar, we do not go into detail.

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[10] Łukasiewicz and Tarski, *Comptes rendus de la Société des Sciences et des Lettres de Varsovie*, Classe III (1930), pp. 30-50; Tarski, *Fundamenta Mathematicae*, vol. 25 (1935), pp. 503-526, especially p. 506.

# A CASE OF COLORATION IN THE FOUR COLOR PROBLEM.*

By C. E. Winn.

In studying the four color problem we assume [1] a map divided by a connected trihedral network into a finite number of polygons. Errera [2] has shown that, if no polygon has more than 6 sides, such a map is reducible, i. e. its coloration can be made to depend on that of one or more maps of fewer polygons. In his treatment, however, the reduced figures are not generally maps of the original type, as they may contain polygons of more than 6 sides. Consequently, the resulting map cannot be further reduced by the same method.

In the present paper we shall obtain reductions or sets of reductions which preserve the type of map in the reduced figure, with a view to proving that

I. *A map S containing at most one polygon of more than* 6 *sides can be colored.*

We shall start by obtaining new reductions which, in conjunction with those already known, may be embodied in the result

II. *Any polygon of less than* 7 *sides in an irreducible map must touch a polygon of more than* 6 *sides.*

We now quote the known reductions required here, giving an explanation of how the reduced figures are formed.

A. *A polygon of less than* 5 *sides.*[3]

The reduction is made by removing any side; only, in the case of a

* Received March 31, 1937.

[1] For a general account of the subject see Sainte-Laguë, "Géométrie de situation et jeux," *Mémorial des Sciences Mathématiques*, fasc. 41, and the thesis of M. A. Errera, "Du coloriage des cartes, etc.," Bothy, Ixelles, 1921.

[2] "Une contribution au problème des quatre couleurs," *Bulletin de la Société Mathématique de France*, vol. 53 (1925), p. 42.

[3] A. B. Kempe, "On the geographical problem of the four colors," *American Journal of Mathematics*, vol. 2 (1879), p. 198.

quadrilateral in a 2-ring,[4] we must remove a side bounding the ring, in order to avoid the creation of an isthmus.[5]

It will be seen at once that the reduction of a digon and a triangle diminishes the vertices of the adjacent polygons by two and one respectively, while that of a quadrilateral deprives of one vertex the two polygons abutting the suppressed side.

B. *A ring of 5 polygons or fewer enclosing more than one polygon.*

The reduced maps for a 2- or 3-ring are formed by suppressing the part of the map on one side of the ring.

Two of the reductions for a 4- or 5-ring are made in the same way; and the others are obtained from them by a further removal of two non-adjacent sides of the quadrilateral or pentagon newly formed (i. e. including more than one polygon of the given map).

The reductions A and B are fundamental in as far as they ensure the presence of two successive rings about each polygon of an irreducible map, without which all other known reductions and those given here might fail, should an isthmus occur in the reduced figure.

C. *A polygon completely surrounded by pentagons.*[6]

In the reduction the polygon coalesces with alternate regions on the further side of the ring, except the last two when the number is odd.

D. *A polygon bounded by hexagons and pairs of pentagons, if not by an odd number of hexagons only.*[7]

In the reduction the whole ring is suppressed except the sides joining the *free* vertices (i. e. belonging to *one* polygon only of the ring) of each hexagon and pair of pentagons.

It may be remarked that the reductions of a pentagon or hexagon flanked

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[4] A ring may be defined as a cyclic sequence of polygons each of which touches that before and after it, but no other one, in the sequence. The polygons of a 2-ring have 2 separate contacts.

[5] An isthmus occurs at a boundary which can be crossed once only by a closed circuit not meeting the network again.

[6] G. D. Birkhoff, "The reducibility of maps," *American Journal of Mathematics*, vol. 35 (1913), p. 116.

[7] Birkhoff, *loc. cit.* (8) and P. Franklin, "The four color problem," *American Journal of Mathematics*, vol. 44 (1922), p. 225.

by 3 pentagons along adjacent sides are not employed here. In fact the latter might involve more than one polygon of 7 sides or more in the reduced figure.

In order to establish II, we must show how to reduce any ring composed of pentagons or hexagons about a pentagon or hexagon. Actually we need only consider those rings in which the pentagons are *isolated.* For, following a remark of Franklin,[8] *we may derive the reduction of a ring containing a pair or pairs of pentagons from that of a ring obtained on replacing them by pairs of hexagons,* assuming every pair reduced as in D. The derived cases are given in brackets at the head of each reduction.

With a view to saving space our reductions are mainly set forth in tabular form referring to the appropriate figure. Under numbered variants are given all essentially distinct groups of the colors 1, 2, 3, 4 bounding the reduced configuration, i. e. which are neither permutable nor symmetrically equivalent. Obvious abbreviations are employed, such as $b = 1$ to mean that the polygon $b$ bears the color 1.

If the solution for any variant is immediate, we mark the color of the central polygon, from which the ring can be filled in without difficulty. Otherwise one or more similar color chains are assumed (1st column), admitting an alternation of the intermediate complementary chain or else a change due to the absence of former chains (2nd column). There are now two possibilities:

(a) The new distribution may lead to a direct coloring (3rd column), in which case we may suppose the assumed chain absent, and replace the previous change by one affecting a polygon named in the first chain, at the same time allowing for consequent modifications elsewhere (4th column). If a direct solution is not yet available, we may assume another chain in the new distribution, and so on until a coloring is finally obtained.

(b) The change first made may not yield a direct coloring. We then examine one or more chains occurring in the derived scheme, continuing as in (a). The digression is distinguished from the main case by means of brackets, which are closed as soon as a solution is forthcoming. In fact *a completed bracket is tantamount to a direct coloring of the previous scheme.* In complicated cases it may be necessary to insert more than one bracket in succession.

[8] *Loc. cit* (8), p. 232.

E. $n5665$ $(n > 5)$. See fig. 1.

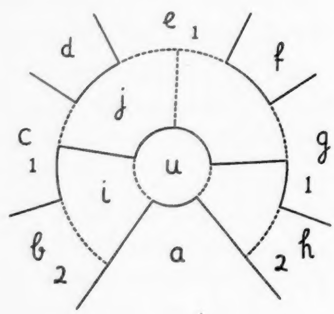


Fig. 1.

Note that, if $a$ were a pentagon, an isthmus would occur in the reduction.

(1) $a = 1$, $d \neq f$ or $d = f = 2$ $u = 2$

(2) $a = 1$, $d = f = 3$

| | | | |
|---|---|---|---|
| 23 $b$ to $f$ | $c = e = 4$ | $u = 4$<br>$u = 3$ | $b = 3$; $d, h = 2$ or 3 |

(3) $a = 3$, $d \neq f$ or $d = f = 4$ $u = 1$

(4) $a = 3$, $d = f = 2$

| | | | |
|---|---|---|---|
| 24 $b$ to $h$ or $d$ | $a = 1$ or $c = 3$ | $u = 2$<br>$u = 1$ | $b = 4$, $f = 2$ or 4 |

(5) $a = 3$, $d = f = 3$

| | | | |
|---|---|---|---|
| 24 $b$ to $h$ | $a = 1$ | (1)<br>$u = 1$ | $b = 4$ |

F. 56666 (or 56655 or 56556). See fig. 2.

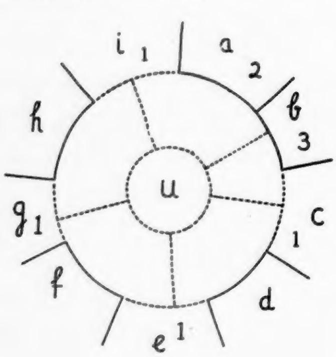


Fig. 2.

(1) $d = 2$ $\qquad$ $u = 1$

(2) $d = 4,\ fh \neq 43$ $\qquad$ $u = 1$

(3) $d = 4,\ f = 4,\ h = 3$

| | | | |
|---|---|---|---|
| 42 $d$ to $a$ | $bc = 13$ | $u = 1$ | $d = 2$ |
| | | $u = 1$ | |

(4) $d = 3,\ f \neq h$ or $f = h = 4$ $\qquad$ $u = 1$

(5) $d = 3,\ f = h = 3$

| | | | |
|---|---|---|---|
| 32 $d$ to $b$ | $c = 4$ | $u = 3$ | $d = 2$ |
| | | $u = 1$ | |

(6) $d = 3,\ f = h = 2$

| | | | |
|---|---|---|---|
| 42 $f$ to $a$ | $g = i = 3$ | | |
| 32 $d$ to $b$ | $c = 4$ | $u = 3$ | $d = 2$ |
| 42 $f$ to $d$ or $a$ | $e = 3, i = 1$ or | $u = 3$ | $f = 4;\ d, h = 2$ or |
| | | $u = 3$ | $f = 2,\ h = 2$ or 4 |
| | | $u = 1$ | |

The next reduction is due to Mr. Choinacki.

G. 66666. See fig. 3.

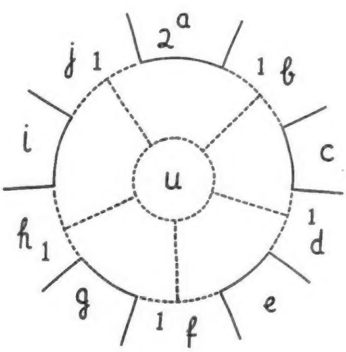


Fig. 3.

(1) cegi not all[9] 2 $\qquad$ $u = 1$

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[9] Cp. the reduction of 666666 about a hexagon by Birkhoff, *loc. cit.* [6].

(2) $c = e = g = i = 2$

| | | | |
|---|---|---|---|
| 24 $a$ to $e$ | $b = d = 3$ | | |
| [23 $i$ to $a$ | $j = 4$ | | |
| {24 $e$ to $c$ or $g$ | $d = 1$ or $f = 3$ | $u = 2$ | |
| 24 $e$ to $a$ | $b = d = 1$, | | |
| | $c = g = 4$ | | |
| (23 $a$ to $i$ | $j = 1$ | $u = 1$ | $a = 3$, $e = 2$ or 3 |
| | | $u = 1$) | $e = 4$ |
| 43 $j$ to $b$ or $d$ | $a = 1$, $c = 2$ or 1 | $u = 1$ | $j = 3$ |
| 24 $i$ to $g$ or $e$ | $h = 3$, $f = 1$ or 3 | $u = 3$ | |
| 24 $i$ to $c$ | $b = j = 1$ | $u = 1$ | $i = 4$, $a = 2$ or 4 |
| | | $u = 3$} | $i = g = 3$ [10] |
| 24 $e$ to $a$ | $d = b = 1$ | $u = 1$ | |
| 24 $e$ to $c$ | $d = 1$ | | |
| (31 $b$ to $d$ or $j$ | $c$ or $a = 4$ | $u = 1$ | $b = 1$ |
| | | $u = 1$) | $e = 4$ |
| | | $u = 3$] | $a = 4$; $c, g, i = 2$ or 4 |
| | | $u = 1$ | |

H. 566666 (or 566655 or 565555). See fig. 4.

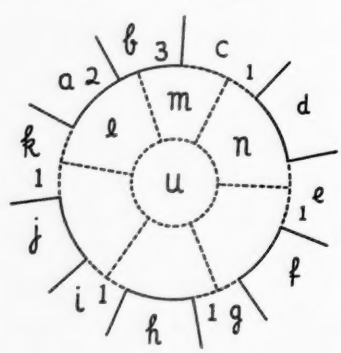


Fig. 4.

(1) If $d = 2$, we get a direct coloring with $u = 1$.

(2) If $d = 4$, we may suppose the 42 chain from $d$ to $a$ present. Hence we have 423 or 342 for $lmn$, directly or after inverting 31 of $bc$. One of these yields a solution with $u = 1$, whatever be the colors of $f$, $h$, $j$.

[10] By symmetry the chain 23 from $e$ to $g$ is also absent.

(3) If $d = 3$, the coloring is direct, provided $f = 4$. And, when $f = 2$, the presence of a 24 chain from $a$ to $f$ allows the inversion $bcde = 1313$. Again one of the alternatives affords a solution for any colors of $h$ and $j$.

(4) If $d = f = 3$, we get a direct coloring for $h = 2$ or 3. When $h = 4$, we proceed as in (2).

The above method can be extended to reduce any polygon enclosed by an *odd* number of hexagons and a pentagon.

The next two reductions are much simplified if we reduce symmetrically by joining the free vertices of the two pentagons across the ring. Unfortunately, however, the two new polygons resulting may both have more than 6 sides, even after A is applied.

J. 565666 (or 565556). See fig. 5.

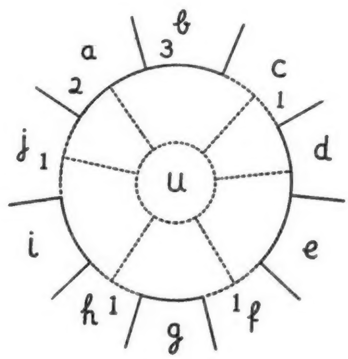


Fig. 5.

(1) $d = 2,\ e = 3,\ gi \neq 34$ $\qquad u = 1$

(2) $d = 2,\ e = 3,\ g = 3,\ i = 4$

| | | | |
|---|---|---|---|
| 12 $j$ to $h$ or $f$ | $i = 3, g = 3$ or 4 | $u = 1$ | |
| 12 $j$ to $c$ | $b = 4, h = f = 2$ | $u = 2$ | $ja = 21$ |
| | | $u = 1$ | |

(3) $d = 2,\ e = 4$

| | | | |
|---|---|---|---|
| 43 $e$ to $b$ | $cd = 21$ | $u = 1$ or 2 | $e = 3$ |
| | | (1) or (2) | |

(4) $d = 3,\ e = 2,\ gi \neq 34$ $\qquad u = 1$

(5) $d=3,\ e=2,\ g=3,\ i=4$

| | | | |
|---|---|---|---|
| 43 $i$ to $b$ | $ja=21$ | | |
| (24 $e$ to $i$ | $fgh=313$ | $u=3$ | $e=4$ |
| | | $u=1$) | $i=3;\ d, g=3$ or 4 |
| | | $u=1$ | |

(6) $d=4,\ e=2,\ g\neq i$ or $g=i=3$ $u=1$

(7) $d=4,\ e=2,\ g=i=2$

| | | | |
|---|---|---|---|
| 23 $e$ to $g$ or $b$ | $f=4$ or $cd=41$ | $u=2$ | $e=3,\ i=2$ or 3 |
| | | $u=1$ | |

(8) $d=4,\ e=2,\ g=i=4$

| | | | |
|---|---|---|---|
| 43 $i$ to $b$ | $ja=21$ | | |
| (42 $d$ to $j$ | $abc=313$ | $u=4$ | $de=24,\ g=4$ or 2 |
| | | $u=1$) | $i=3;\ d, g=4$ or 3 |
| | | $u=1$ | |

(9) $d=4,\ e=3,\ gi\neq 32$ $u=1$

(10) $d=4,\ e=3,\ g=3,\ i=2$

| | | | |
|---|---|---|---|
| 32 $e$ to $b$ | $cd=41$ | | |
| (42 $c$ to $a$ | $b=1$ | $u=1$ | $c=2,\ i=2$ or 4 |
| | | $u=1$) | $e=2;\ g, i=2$ or 3 |
| | | (6) or (7) | |

(11) $d=3,\ e=4,\ g\neq i$ or $g=i=3$

| | | | |
|---|---|---|---|
| 42 $e$ to $a$ | $bcd=131$ | $u=1$ | $e=2$ |
| | | (4) or (5) | |

(12) $d=3,\ e=4,\ g=i=2$

| | | | |
|---|---|---|---|
| 42 $e$ to $a$ | $bcd=131$ | | |
| (23 $a$ to $c$ | $b=4$ | $u=2$ | $a=3;\ g, i=2$ or 3 |
| | | $u=1$) | $e=2;\ g, i=2$ or 4 |
| | | $u=1$ | |

(13) $d = 3,\ e = 4,\ g = i = 4$

| | | | |
|---|---|---|---|
| 43 $i$ to $b$ | $ja = 21$ | | |
| (14 $a$ or $c$ to $i$ | $j = 3,\ b = 2$ or 3 | $u = 1$ | $a = c = 4$ |
| 43 $i$ to $a, g$ or $e$ | $j = 1,\ h = 2$ or | | |
| | $h = f = 2$ | $u = 4$ | $i = 3$ |
| 32 $d$ to $b$ or $j$ | $c = 1,\ a = 4$ or 1 | $u = 1$ | $d = 2$ |
| 34 $i$ to $e$ | $h = f = 2$ | $u = 4$ | $i = 4;\ a, g = 4$ or 3 |
| | | $u = 4$) | $i = 3$ |
| | | (9) or (11) | |

K. 566566. See fig. 6.

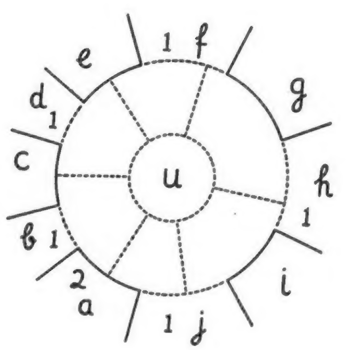


Fig. 6.

(1) $c = e = 2$ $\quad u = 1$

(2) $c = 2,\ e = 3,\ gi \neq 42$ $\quad u = 1$

(3) $c = 2,\ e = 3,\ g = 4,\ i = 2$

| | | | |
|---|---|---|---|
| 32 $e$ to $c$ | $d = 4$ | $u = 2$ | $e = 2;\ a, i = 2$ or 3 |
| | | $u = 1$ | |

(4) $c = 3,\ e = 4,\ g \neq i$ or $g = i = 3$ $\quad u = 1$

(5) $c=3,\ e=4,\ g=i=2$ ($g=i=4$ equivalent)

| | | | |
|---|---|---|---|
| 43 $e$ to $c$ | $d=2$ | | |
| [42 $e$ to $a$ | $bc=31$ | $u=1$ | |
| 42 $e$ to $i$ | $f=h=3$ | | |
| {23 $a$ to $i$ | $j=4$ | | |
| (43 $j$ to $c, h, f$ | $ab=12,\ i=1,$ | | |
| | $g=i=1$ | $u=3$ | $j=3$ |
| | | $u=3$) | $a=3,\ c=3$ or 2 |
| | | $u=3$} | $e=2,\ g=2$ or 4 |
| | | $u=2$] | $e=3$ |
| | equivalent of | (2) or (3) | |

(6) $c=3,\ e=2,\ g\neq i$ or $g=i=2$

| | | |
|---|---|---|
| 23 $e$ (or $a$) to $c$ $d$ (or $b$) $=4$ | $u=2$ | $c=2$ |
| | (1) | |

(7) $c=3,\ e=2,\ g=i=3$

| | | | |
|---|---|---|---|
| 24 $e$ to $a$ | $bcd=313$ | | |
| [34 $b$ to $d$ | $c=2$ | | |
| {31 $d$ to $f$ | $e=4$ | | |
| (34 $d$ to $b$ or $i$ | $c=1;\ a=2$ or 1 | $u=1$ | |
| 34 $d$ to $g$ | $de=43,\ g=4$ | $u=2$ | $de=43$ |
| 32 $e$ to $c$ | $d=1$ | $u=3$ | $e=2;\ g, i=2$ or 3 |
| | | $u=1$) | $d=1,\ b=1$ or 3 |
| | | $u=1$} | |
| 34 $b$ to $g$ | $d=4$ | $u=1$ | $b=4,\ i=3$ or 4 |
| | | $u=1$] | $e=4$ |
| | | $u=1$ | |

(8) $c=3,\ e=2,\ g=i=4$

| | | | |
|---|---|---|---|
| 42 $e$ to $a$ | $bcd=313$ | | |
| {43 $b$ to $i$ | $aj=12$ | $u=1$ | |
| ($d$ to $g$) | | | |
| 43 $b$ to $d$ | $b=d=4$ | | |
| (41 $b$ to $j$ | $e=3$ | $u=1$ | $bcd=141$ |
| | equivalent to (7)) | | $b=4,\ g=3$ or 4 |
| | | $u=1$} | $e=4;\ g, i=4$ or 2 |
| | | (4) or (5) | |

L. 565656. See fig. 7.

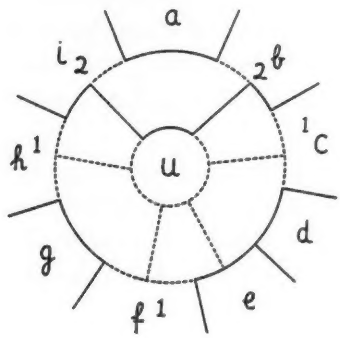


Fig. 7.

(1) $a = 1,\ deg \neq 234$ $\qquad u = 1$

(2) $a = 1,\ d = 2,\ e = 3,\ g = 4$

| | | | |
|---|---|---|---|
| 24 $d$ to $b$ or $i$ | $c = 3,\ a = 1$ or 3 | $u = 2$ | $d = 4,\ g = 4$ or 2 |
| | | $u = 1$ | |

(3) $a = 3,\ d = 3$ $\qquad u = 1$

(4) $a = 3,\ de = 32,\ g \neq 4$ $\qquad u = 1$

(5) $a = 3,\ de = 23,\ g = 4$

| | | | |
|---|---|---|---|
| 23 $d$ to $b$ | $c = 4$ | $u = 2$ | $d = 3$ |
| | | $u = 1$ | |

(6) $a = 3,\ de = 24$

| | | | |
|---|---|---|---|
| 23 $d$ to $b$ | $c = 4$ | $u = 2$ or 4 | $d = 3$ |
| | | $u = 1$ | |

(7) $a = 3,\ de = 42,\ g \neq 3$ $\qquad u = 2$

(8) $a = 3,\ d = 4,\ e = 2,\ g = 3$

| | | | |
|---|---|---|---|
| 34 $g$ to $d$ | $ef = 12$ | $u = 1$ | $g = 4,\ a = 3$ or 4 |
| | | $u = 2$ | |

(9) $a = 3,\ de = 43,\ g \neq 2$ $\qquad u = 1$

(10) $a = 3,\ de = 43,\ g = 2$

| | | | |
|---|---|---|---|
| 42 $d$ to $b$ or $i$ | $c = 3,\ a = 1$ or 3 | $u = 2$ | $d = 2,\ g = 2$ or 4 |
| | | (4) or (5) | |

We proceed to color a map $S$ which may contain one polygon $x$ of more than 6 sides—otherwise we let $x$ be any polygon of $S$.

First we apply the reduction A to any polygon of less than 5 sides. The removal of any side plainly leaves us with a map of type $S$; and an exhaustion of the process leads either to a map of 3 digons, which is colorable, or to one free of A polygons.

Next, B is used in conjunction, if necessary, with A. The suppression of the configuration flanking a B ring again leaves a map of type $S$, as the only new polygon has at most 5 sides. It remains for us to examine the other reductions already explained for a 4 or 5-ring, concerning which we shall justify the following assumptions:

(1) Any 3 polygons $abc$ of a 4-ring $R$ have at least 2 free vertices on either side of $R$, and therefore at most 4 when $R$ is composed of pentagons and hexagons only.

For, if $abc$ have one vertex only on one side of $R$, the fourth polygon $d$ has one or more vertices on this side. In the former case $R$ encloses 2 polygons with a total of 8 vertices (unless they form a 2-ring). In the latter case we get a 3-ring with $d$. Further, if $abc$ have no vertex on one side of $R$, they touch a quadrilateral or a polygon making a 2-ring with $d$. Consequently, one of the previous reductions is applicable.

(2) Any 4 polygons of a $B$ 5-ring $R$ have together at least 3 vertices on either side of $R$, and so 5 at most when $R$ is composed of pentagons and hexagons only.

For, if not, we should get in the same way a ring of 4 or less, or else 2 polygons with a total of 9 vertices or 3 with a total of 13 or 14, according as all or only two of the latter touch the fifth polygon of $R$.

(3) If $x$ occurs in a $B$ 5-ring at $a$, the other polygons $bcde$ possess at most 4 free vertices on either side of $R$, unless both $b$ and $e$ have a free vertex on each side of $R$.

Otherwise, we can replace $b$ or $e$ by the polygon touching $abc$ or $dea$. It is easily verified from a figure that the new ring has at most 4 vertices on either side, and on account of (2) surrounds more than one polygon.

Now consider the reduction of a 4-ring composed of $abcd$ which is made by uniting $a$ and $c$ into a new polygon $y$. If $a$ or $c \equiv x$, there is clearly no gain of vertices except for $x$. Otherwise, if $x$ still occurs in $R$, let $b \equiv x$.

We see that $y = 5$ or 6 unless $a$, $c$ have 3 or 4 vertices outside $R$ (i. e. on the unreduced side), in which case $y = 7$ or 8. In view of (1), however,

$d$ then possesses one or no free vertex outside $R$, and so becomes a triangle or digon on the further reduction of which we get $y = 6$, as required.

Let us pass to the other reductions of a $B$ ring composed of $abcde$, first supposing $a \equiv x$. As the fusion of $a$ with $c$ or $d$ will yield no gain of vertices elsewhere, we need only deal with the two figures formed by uniting $bd$ (or $ce$) and $be$.

In the former case the new polygon $y$ will have $r + 5$ vertices, where $b$ and $d$ have together $r$ free vertices outside $R$. Thus $y > 6$ only when $r = 2$, 3 or 4.

When $r = 2$, we can further reduce $c$ $(< 5)$ so as to diminish[11] the vertices of $y$ to less than 7.

When $r = 3$, since $c$, $e$ have, according to (2), at most 2 free vertices outside $R$, either $c < 5$, $e < 5$ or $c = 2$. In each case their further reduction brings down the number of vertices of $y$ by 2 to less than 7.

When $r = 4$, it follows from (3) that $c$, $e$ have no free vertices outside $R$, so that their further reduction diminishes the vertices of $y$ by 3 to less than 7.

The proof for the reduction of a 5-ring in which $b$ and $e$ are united, runs on similar lines, use being made of the further reduction of $c$ and $d$, when necessary.

Finally, if $x$ does not occur in $R$, we can no longer appeal to (3). But the above proof holds for $r = 2$ or 3 without change, while the further reduction of $a$ yields the required result when $r = 4$. Moreover, as no polygon of $R$ has more than 6 sides, a single reduction suffices, the other cases being derived by symmetrical or cyclic interchange.

When the reductions A and B are no longer possible, we obtain a map of 3 digons or one of type $S$ for which any of the other reductions quoted or proved are available without the danger of an isthmus appearing in the reduced figure. In each case we shall see that the resulting map is of type $S$. We apply these reductions in the following order:

(1) D, E, F, G, H, J or K to a polygon $u$ connected with $x$ by the common side of two hexagons. In each reduction the unreduced sides of the two hexagons are separate, so that, except in E, *$u$ and $x$ form a single polygon.* There is consequently no gain of vertices elsewhere.

(2) D, E, F, H, J, K or L to a polygon connected with $x$ by the common

[11] If $c$ has 4 sides left, the previous reductions of 2-rings obviates the creation of an isthmus on the further reduction of $c$. The same is true of other quadrilaterals to be reduced later.

side of a hexagon and a pentagon. By employing one of the reduced figures or its symmetrical or cyclic equivalent we again ensure that $u$ and $x$ be united in the reduction, except possibly in the case E. Thus in F, if $b \equiv x$, we join the free vertex of the pentagon to those of the other adjacent hexagon. Similarly, a cyclic adjustment of fig. 7 for L may be necessary to make $x$ coalesce with $u$, the other new polygon having at most 6 vertices.

As regards E, if $c$, $e$ or $g \equiv x$, then $a$, being a hexagon, is reduced to a digon. Consequently, the new polygon comprising $u$, which has not more than 8 vertices, retains at most 6 when the digon is reduced. Again, if $b$ (or $h$) $\equiv x$ and $c = 5$, the new polygon including $c$ has at most 8 vertices, which we can diminish to 6 or less by further reducing $d$ or $f$. On the other hand, if $b \equiv x$ and $c = 6$, we apply instead D or H to the hexagon $j$, which touches $c$ and $i$ in common with $x$.

(3) C to $x$, when surrounded by pentagons only. The reduction is seen to produce no gain of vertices except for $x$.

Since one or other of the above configurations is always present, we have shown how to reduce $S$ to a map of the same type, and hence to obtain the required coloration.

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# ON THE FUNDAMENTAL GROUP OF A CERTAIN CLASS OF PLANE ALGEBRAIC CURVES.*

By W. S. Turpin.

**1. Introduction.** The problem of existence of algebraic functions, $z$, of two independent variables, $x$ and $y$, possessing a preassigned branch curve of order $n$

$$f_n(x,y)=0 \tag{1}$$

has been considered by Enriques [1] and Zariski.[2] Zariski has shown that, in view of a result of Enriques, this question may be reduced to the consideration of the Poincaré (fundamental) group of the residual space of the branch curve (1) relative to its carrying complex projective plane and the application of the Riemann existence theorem for algebraic functions of one variable having preassigned branch points.

It is sufficient for the theory of algebraic surfaces from the point of view of birational transformations to consider branch curves (1) possessing only ordinary double points and cusps. Zariski has shown that if a curve possesses only ordinary double points then its fundamental group is necessarily cyclic. A simple case of a curve whose fundamental group is not cyclic is that of the branch curve of a cubic surface. If the cubic surface is general, its branch curve is a sextic, $f_6$, with six cusps on a conic:

$$f_6\,:\,[\phi_3(x,y)]^2+[\psi_2(x,y)]^3=0, \tag{2}$$

where $\phi_3$ and $\psi_2$ are polynomials in $x$ and $y$ of respective degrees 3 and 2. This curve was treated in detail by Zariski and its fundamental group specifically determined.

An obvious generalization of the curve (2) is the curve $f_{6m}$, of order $6m$, with $6m$ cusps at the intersections of two curves, $\phi_{3m}(x,y)=0$ and $\psi_{2m}(x,y)=0$, of orders $3m$ and $2m$ respectively, where $m$ is a positive integer. Such a curve is given by the equation:

$$f_{6m}\,:\,[\phi_{3m}(x,y)]^2+[\psi_{2m}(x,y)]^3=0. \tag{3}$$

* Received December 1, 1936.

[1] F. Enriques, "Sulla costruzione delle funzioni algebriche di due variabili possedenti una data curva di diramazione," *Annali di matematica pura ed applicata*, ser. 4, vol. 1 (Nov., 1923), pp. 185-198.

[2] O. Zariski, "On the problem of the existence of algebraic functions," *American Journal of Mathematics*, vol. 51 (1929), pp. 305-328.

The methods of investigation used for the fundamental group of (2) are peculiar to the sextic of this type and do not admit of an extension to the more general class of curves (3). Hence, it was deemed of interest to investigate the structure of the fundamental group of curves of the type (3). One may expect that the methods developed in this investigation may point the way to a possible procedure for other types of curves, for instance for the branch curves of general surfaces of any order in $S_3$. As it is known, these curves have been completely characterized by B. Segre.[3]

This investigation of the structure of the fundamental group falls under three classifications:

1°. The determination of the fundamental group, $\bar{G}$, of a degenerate limit curve, $\bar{f}$, of curves (3).

2°. The factorization of the relations of $\bar{G}$ into relations belonging formally to the fundamental group $G'$ of a virtual curve $f'$ with $6m^2$ cusps, of which $\bar{f}$ is a limit curve.

3°. Verification of the fact that $G' = G$, where $G$ denotes the fundamental group of a curve $f$ of type (3).

Our method of attacking the problem in our special case contains the nucleus of a perfectly general procedure, applicable to an arbitrary plane curve with nodes and cusps, provided the complete continuous (irreducible) system $\{f\}$ of curves having the same singularities as $f$ contains some special curve $\bar{f}$, for instance a degenerate curve without multiple components, whose fundamental group can be directly determined. However, to date, a method has not been found for the step 3° of the above procedure that does not appeal to the special geometry of the curves $f$. This verification is necessary due to the fact that the factorization obtained in 2° is not unique. It seems probable that an equivalence of possible factorizations can be established by purely group theoretic considerations, but, efforts in this direction have not been successful up to the present.

**2. General properties of the fundamental group $G$ and its associated group $T$.**[4] Consider the curve $f$ determined by the equation $f(x, y) = 0$ where $f(x, y)$ is a polynomial, of degree $n$, in the complex variables $x$ and $y$.

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[3] B. Segre, "Sulla caratterizzazione delle curve di diramazione nei piani multipli generali," *Mem. Accad. Ital.*, Mat., vol. 1 (1930).

[4] The concepts and results in this section are compiled from the following sources: S. Lefschetz, "Topology," *American Mathematical Society Colloquium Publications*, vol. 12 (1930); E. R. van Kampen, "On the fundamental group of an algebraic curve," *American Journal of Mathematics*, vol. 55 (1933); O. Veblen, "Analysis situs," *American Mathematical Society Colloquium Publications*, vol. 5 (1922); O. Zariski, "On the

We shall be interested in the Poincaré group, $G$, of the residual space, $S$, of the curve $f$ relative to its carrying complex projective plane $(x, y)$.

Let us suppose that the coördinate axes have been chosen in such a manner that $f$ does not pass through the point at infinity on the $y$-axis and that it possesses no multiple components. A generic line of the pencil $\{x = \text{const.}\}$ will thus have $n$ distinct intersections with the curve $f$. Moreover, lines of this pencil having less than $n$ distinct intersections with $f$ are finite in number. We shall call such lines singular lines of the pencil and denote them by $x = \alpha_i$ $(i = 1, 2, \cdots, \nu)$. Denote by $\alpha_0$ a point distinct from the set $[\alpha_i]_{i=1, 2, \ldots, \nu}$ and let $[\delta'_i]_{i=1, 2, \ldots, \nu}$ be a set of non-intersecting loops in the plane of the variable $x$ emanating from $x = \alpha_0$ and surrounding respectively the points of the set $[\alpha_i]_{i=1, 2, \ldots, \nu}$. Let $y_k = b_k$ $(k = 1, 2, \cdots, n)$ be the roots of $f(\alpha_0, y) = 0$ and choose loops $g_k$ in the plane of the complex variable $y$, emanating from the point at infinity and surrounding $b_k$, in such a manner that $g_i$ and $g_j$ have only the point at infinity in common for $i \neq j$. We denote this base point of the loops $g_k$ by $O$.

If we define the Poincaré group $G$ of $f$ in the usual manner, it is well known that the loops $g_k$ may be taken as generators of $G$ and that $G$ is independent of the choice of $O$ to within an isomorphism. It is evident that the generators $g_k$ satisfy the following relation:

$$g_1 g_2 \cdots g_n = 1. \tag{4}$$

As $x$ traverses a closed path in its plane, starting from and returning to $\alpha_0$, the set of points $[b_k]$ will move continuously describing certain paths in the $y$-plane and returning finally to its original position, although the individual points may have been permuted among themselves. As the points $b_k$ move, their corresponding loops $g_k$ will also vary and this variation is completely determined by the motion of the points $b_k$ and the condition that the set $[g_k]$ should always consist of non-intersecting loops.

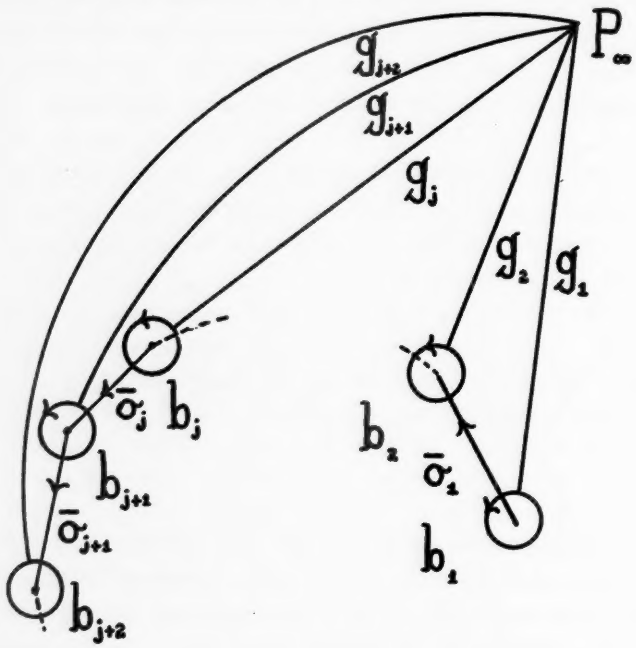
If, under cyclical variation of $x$ from $\alpha_0$, a root $y$ traverses a path from $b_i$ to $b_j$, the loop $g_i$ is transformed into a loop $g'_i$ surrounding $b_j$ alone and which must therefore be a transform of $g_j$ by some element of $G$. Moreover, $g_i$ and $g'_i$ are equivalent elements of $G$ and, thus, corresponding to every cyclical variation of $x$ from $\alpha_0$, we obtain a relation between the elements of $G$. In particular, if $x$ traverses the loops $[\delta'_i]_{i=1, \ldots, \nu}$ we obtain motions of the roots $y_k$ which yield the relations

problem of existence of algebraic functions," *loc. cit.*; O. Zariski, "On the Poincaré group of rational plane curves." *American Journal of Mathematics*, vol. 58 (1936).

$$(5) \qquad \Phi_{k,i}(g_1, g_2, \cdots, g_n) \equiv g_k^{-1} g_{k,i}^{-1} g_{p_{ki}} g_{ki} = 1 \qquad \begin{pmatrix} k = 1, 2, \cdots, n \\ i = 1, 2, \cdots, \nu \end{pmatrix}$$

where $\{p_{1,i}, p_{2,i}, \cdots, p_{n,i}\}$ is a permutation of $\{1, 2, \cdots, n\}$. It has been shown that the relations (5) together with the relation (4) constitute a complete set of generating relations for the group $G$.

The class of motions of the set of points $[b_k]$ induced by cyclic variation of $x$ incident to the point $\alpha_0$ which carry this set into its original position,




y:plane

Fig. 1.

in such a way that the roots $y_k$ remain distinct during the motion, constitutes a group of motions $T$. Two such motions, $m_1$ and $m_2$, correspond to the same element of $T$ if the motion $m_1$ may be deformed into the motion $m_2$ by suitable deformation of the path of $x$ so that, during the latter deformation, the induced motions keep the roots $y_k$ distinct.

Let $\bar{\sigma}_j$ $(j = 1, 2, \cdots, n-1)$ denote an oriented arc from $b_j$ to $b_{j+1}$ of such a type that the adjacent arcs have only an end point in common and non-adjacent arcs do not intersect (see Fig. 1). Let $T_j$ $(j = 1, 2, \cdots, n-1)$

denote a motion in which the points $b_i$ $(i \neq j, j+1)$ are fixed while the points $b_j$, $b_{j+1}$ are interchanged, $y_j$ moving from $b_j$ to $b_{j+1}$ along the right-hand edge of $\bar{\sigma}_j$ and $y_{j+1}$ moving from $b_{j+1}$ to $b_j$ along the opposite edge. Any motion, $m$, belonging to $T$ can be deformed into a motion, $m'$, which may be expressed as a product of the elementary motions $T_j$. The deformation of $m$ into $m'$ affects not only the paths but also the velocities of the points $b_k$. The elementary motions $T_j$ satisfy the relations:

$$T_iT_j = T_jT_i, \qquad |\,i-j\,| \neq 1 \tag{6}$$

$$T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}. \tag{7}$$

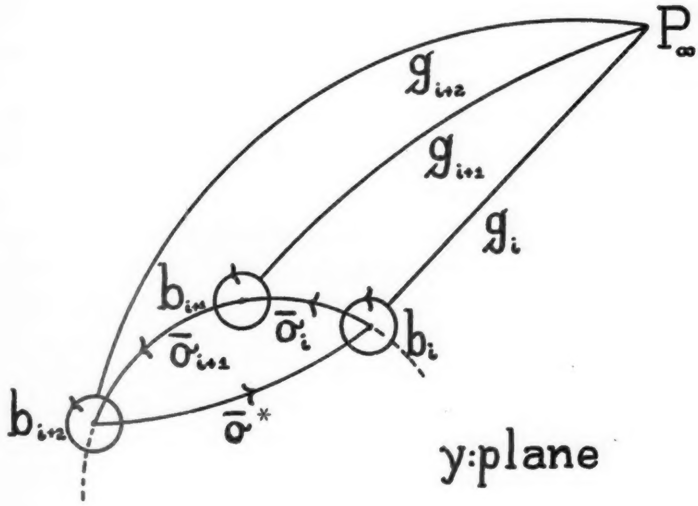



Fig. 2.

The commutative relation, (6), holds due to the fact that, under the assumption $|\,i-j\,| \neq 1$, $\bar{\sigma}_i$ and $\bar{\sigma}_j$ have nothing in common and the motions $T_i$ and $T_j$ are thus independent of the order of their performance. The proof of (7) is as follows: Introduce the motion $T^*$ sending $y_{i+2}$ from $b_{i+2}$ to $b_i$ along the right-hand edge of the oriented arc $\bar{\sigma}^*$ and $y_i$ from $b_i$ to $b_{i+2}$ along its opposite edge, where $\bar{\sigma}^*$ is chosen as indicated in Fig. 2. Then the following equalities hold; $T^*T_{i+1} = T_{i+1}T_i = T_iT^*$. Solving these for $T^*$ and equating the solutions, we have that $T_{i+1}T_iT_{i+1}^{-1} = T_i^{-1}T_{i+1}T_i$. Multiplying this relation on the right by $T_{i+1}$ and on the left by $T_i$, we obtain the desired result.

If we now choose $g_1$ to be any loop on $O$ surrounding $b_1$ which does not intersect the arcs $\bar{\sigma}_2, \cdots, \bar{\sigma}_n$, and let $g_i$ be the loop into which $g_{i-1}$ is deformed as $y_{i-1}$ moves from $b_{i-1}$ to $b_i$ along the arc $\bar{\sigma}_{i-1}$ for $(i = 2, \cdots, n)$, then the

motion $T_j$ induces a transformation of the following type on the set of loops $[g_k]$:

$$(8) \qquad t_j : \begin{cases} g'_i = g_i, & (i \neq j, j+1) \\ g'_j = g_{j+1} \\ g'_{j+1} = g_{j+1}^{-1} g_j g_{j+1} & (j = 1, 2, \cdots, n-1). \end{cases}$$

If we denote the complement of the set of points $[\alpha_i]_{i=1,2,\ldots,\nu}$ relative to the plane of the complex variable $x$ by $C[\alpha_i]$, then, as $x$ describes a cyclic path incident to $\alpha_0$ in $C[\alpha_i]$, the points $[b_k]$ undergo a motion belonging to the group $T$, the loops $g_k$ are subjected to the corresponding transformation which, in turn, yields a generating relation of the group $G$. In particular, the motions $T_{\delta'_i}$ generated as $x$ describes the loops $\delta'_i$ induce transformations which yield the relations (5) for $G$. In consequence, a knowledge of the motions $T_{\delta'_i}$ is sufficient to determine the structure of $G$.

It will be useful to examine the motion induced on the roots $y$ as $x$ describes a loop about a singular value corresponding respectively to a tangent, ordinary double point and cusp of $f$.

(A) Suppose $x = \alpha_i$ is a simple tangent to the curve $f$ and that $y_1 = b_1$, $y_2 = b_2$ are the two roots of $f(\alpha_0, y) = 0$ which tend towards coincidence as $x \rightarrow \alpha_i$. Then, as $x$ describes $\delta'_i$, the motion of the roots $y_k$ is equivalent to one in which $y_j$ is fixed for $j > 2$ and the motion of $y_1$ and $y_2$ is typified by that of the roots of $y^2 = x$ as $x$ describes the loop $|x| = 1$. This motion has the form $T_1$. If we do not make the simplifying assumption to the effect that the two roots which approach coincidence as $x \rightarrow \alpha_i$ have consecutive indices, the corresponding motion will have the form $\bar{T}^{-1} T_1 \bar{T}$ where $\bar{T}$ is a product of elementary motions $T_j$.

(B) Suppose $x = \alpha_i$ is a singular value corresponding to an orrdinary double point of $f$. Then the motion of the roots induced as $x$ describes $\delta'_i$ is equivalent to $T_1^2$, or, in the general case, to $\bar{T}^{-1} T_1^2 \bar{T}$. Such a singular value $\alpha_i$ may be considered as the limit of two singular values $\alpha_i'$ and $\alpha_i''$, corresponding to simple tangents of $f$, at each of which the same pair of roots is permuted.

(C) Suppose $x = \alpha_i$ is a singular value corresponding to a cusp of $f$. Then the motion of the roots induced as $x$ describes $\delta'_i$ is equivalent to $T_1^3$, or, in the general case, to $\bar{T}^{-1} T_1^3 \bar{T}$. In this case, the singular value $\alpha_i$ can be considered as the limit of three singular values corresponding to simple tangents which have approached coincidence and at all of which the same pair of roots is permuted.

The relations arising between the generators $g_i$ of $G$ due to singular values of types (A), (B) and (C) are, respectively, of the following forms:

(a) $g_1 = g_2$ (b) $g_1 g_2 = g_2 g_1$ (c) $g_1 g_2 g_1 = g_2 g_1 g_2$.

Let us now consider a variable curve $f$ and let $\bar{f}$ denote a limit curve under continuous variation of $f$. If $f$ and $\bar{f}$ have the same singularities, they are isotopic and, accordingly, possess the same fundamental group. Suppose, however, that as $f$ tends towards $\bar{f}$ it acquires new multiple points or multiple points of higher order. For the sake of simplicity and definiteness, let us suppose that as $f \to \bar{f}$ the simple singular values $\alpha_1$ and $\alpha_2$ of $f$ approach coincidence in $\alpha$ so that $\bar{f}$ acquires a new ordinary double point. Any generating relation for $f$ corresponding to a singular value distinct from $\alpha_1$, $\alpha_2$ remains a true relation for $\bar{f}$. The relations for $f$ relative to $\alpha_1$ and $\alpha_2$, viz. $g_1 = g_2$, are destroyed for $\bar{f}$ since, in the limit when $\alpha_1, \alpha_2 \to \alpha$, both the loops $\delta'_1$ and $\delta'_2$ pass through $\alpha$. On the other hand, the relation $g_1 g_2 = g_2 g_1$ for $f$ which arises from a circuit, $\delta'$, of $x$ about both $\alpha_1$ and $\alpha_2$ is not destroyed for $\bar{f}$ since, in the limit, it becomes the relation corresponding to a circuit of $x$ about the singular value $\alpha$ for $\bar{f}$. This reasoning is perfectly general and applies to any new multiple point, or point of higher order, acquired by $\bar{f}$. In fact, the following theorem has been established:

*All generating relations of the fundamental group $\bar{G}$ of a limit curve $\bar{f}$ of a variable curve $f$ are also true generating relations for the fundamental group $G$ of $f$.* This theorem holds even if $\bar{f}$ is degenerate provided that it possesses no multiple components.

**3. Consideration of the motions, $T$, occurring at the singular values of a degenerate case of $f_{6m}(x, y)$.** We wish to investigate the fundamental group of curves of the type

$$f_{6m}(x, y) \equiv [\phi_{3m}(x, y)]^2 + [\psi_{2m}(x, y)]^3 = 0 \tag{3}$$

where $\phi_{3m}$ and $\psi_{2m}$ are polynomials in the two complex variables $x$ and $y$ of respective degrees $3m$ and $2m$. It is supposed that the intersections of $\phi_{3m} = 0$ and $\psi_{2m} = 0$ are distinct. If the curves $\phi_{3m} = 0$ and $\psi_{2m} = 0$ are general, the curve $f_{6m} = 0$ will possess $6m^2$ cusps at the intersections of the curves $\phi_{3m} = 0$ and $\psi_{2m} = 0$ and no further singular points. For a general choice of coördinate axes, $f_{6m}$ will possess, under these restrictions, $6m^2$ distinct critical values corresponding to cusps and $6m(3m - 1)$ distinct critical values corresponding to tangents.

We postpone the consideration of (3) temporarily and consider a degenerate case of a curve of this type, namely,

$$\bar{f}'_{6m}(x, y) \equiv y^{6m} - x^{3m} = 0. \tag{9}$$

The critical values for this function are $x = 0$ and $x = \infty$. The singularity corresponding to each of these values consists of $3m$ branches having simple contact and vertical branch tangent. We now proceed to ascertain the motion

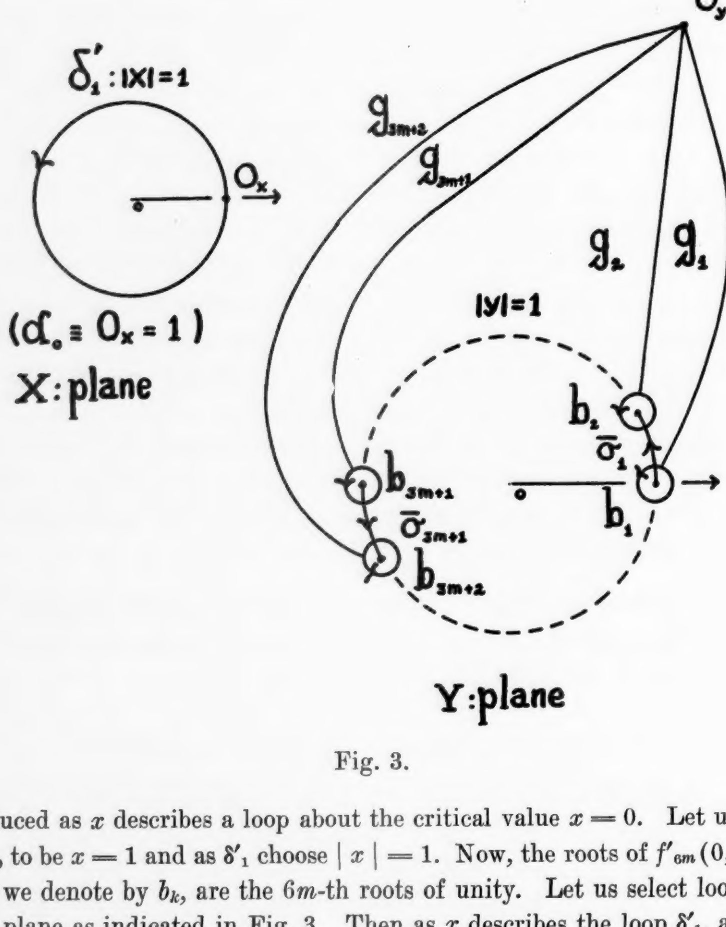



Fig. 3.

$T'$ induced as $x$ describes a loop about the critical value $x = 0$. Let us choose $x = \alpha_0$ to be $x = 1$ and as $\delta'_1$ choose $|x| = 1$. Now, the roots of $f'_{6m}(0, y) = 0$, which we denote by $b_k$, are the $6m$-th roots of unity. Let us select loops $g_i$ in the $y$: plane as indicated in Fig. 3. Then as $x$ describes the loop $\delta'_1$, a motion $T'$ is set up which sends $b_i$ into the diametrically opposite point $b_{3m+i}$ along the arc $\overrightarrow{b_i b_{3m+i}}$. Let $T_i$ denote the elementary motion corresponding to the arc $\bar{\sigma}_i$ of Fig. 3. Then the motion $T'$ may be expressed in terms of the elementary motions $T_i$ as follows:

$$T' = (T_{6m-1}T_{6m-2} \cdot \cdot \cdot T_2T_1)^{3m}. \tag{10}$$

Since the loop $\delta'_1$ may also be considered as a loop about $x = \infty$, the motion of the roots induced as $x$ describes a loop about the latter critical point is again equivalent to $T'$.

**4. Factorization of the motion $T'^2$ into elementary motions belonging to virtual cusps and simple tangents.** The curve (9) is a member of the continuous subsystem of the system $\{\bar{f}_{6m}\}$ consisting of the curves

$$\bar{f}^*_{6m} : y^{6m} = (x - a)^{3m}(x - b)^{3m}, \tag{11}$$

and is isotopic to the general curve of this subsystem. In fact, the equations

$$\begin{aligned} x' &= \frac{x - \tau b}{\tau x + 1 - \tau(1 + a)} \\ y' &= \frac{y}{\tau x + 1 - \tau(1 + a)} \end{aligned} \tag{12}$$

where $\tau$ is a parameter, define for each value of $\tau$ such that

$$\tau b - \tau(1 + a) + 1 \neq 0,$$

a non-degenerate collineation $\pi_\tau$ in the projective $(x, y)$ plane which carries the curve (9) into a curve of the system $\{\bar{f}^*_{6m}\}$. Moreover, $\pi_0$ is the identity while $\pi_1(\bar{f}^*_{6m})$ is the curve (9).

The curve $\bar{f}^*_{6m}$ is a limit curve of the general curve, $f_{6m}$, and we have, in the pencil $\{x = \text{const.}\}$, only two singular lines for $\bar{f}^*_{6m}$, $x = a$ and $x = b$. These singular lines each absorb a certain number of singular lines with respect to the general curve, $f_{6m}$, and since the singular lines $x = a$ and $x = b$ can be interchanged by a continuous variation of the curve $\bar{f}^*_{6m}$ in the system (11), it follows that each of these lines absorbs $3m^2$ lines $x = \text{const.}$ passing through cusps of $f_{6m}$ and $3m(3m - 1)$ simple tangents of $f_{6m}$. It must therefore be possible to factor the motion $T'^2$ into a product of motions $\bar{T}_i$, $6m^2$ of which correspond to cusps and the remaining $6m(3m - 1)$ of which correspond to simple tangents. We proceed to exhibit formally one such factorization of $T'^2$, making use of the relations (6) and (7), and we shall show afterwards that this factorization actually belongs to the curve $\bar{f}^*_{6m}$ considered as a limiting case of the general curve of the system $\{f_{6m}\}$.

We first write $T'^2$ in the form

$$T'^2 = (T_{6m-1}T_{6m-2} \cdot \cdot \cdot T_1)^{6m} = [(T_{6m-1}T_{6m-2} \cdot \cdot \cdot T_1)^2]^{3m}. \tag{13}$$

It will be useful to establish two lemmas.

LEMMA 1. $T_jT_{j-1}T_{j-2}(T_jT_{j-1})^{-1} = (T_{j-1}T_{j-2})^{-1}T_jT_{j-1}T_{j-2}$.

*Proof.* Making use of an alternate form of (7) we have

$$T_jT_{j-1}T_{j-2}T^{-1}_{j-1}T_j^{-1} = T_jT^{-1}_{j-2}T_{j-1}T_{j-2}T_j^{-1}$$

and in virtue of (6)

$$T_jT^{-1}_{j-2}T_{j-1}T_{j-2}T_j^{-1} = T^{-1}_{j-2}T_jT_{j-1}T_j^{-1}T_{j-2}.$$

Repeating the first process, we obtain

$$T^{-1}_{j-2}T_jT_{j-1}T_j^{-1}T_{j-2} = T^{-1}_{j-2}T^{-1}_{j-1}T_jT_{j-1}T_{j-2}$$

which is the desired result.

LEMMA 2. $(T_jT_{j-1})^2 = (T_j^{-1}T_{j-1}T_j)T^3{}_{j-1}$.

*Proof.* Using (7), we may write

$$T_jT_{j-1}T_jT_{j-1} = T_{j-1}T_jT^2{}_{j-1} = (T_{j-1}T_jT^{-1}_{j-1})T^3{}_{j-1}$$

and making use of an alternate form of (7) we are enabled to write the last member in the desired form.

Let us now write $(T_{6m-1}T_{6m-2}\cdots T_1)^2$ in the following manner:

$$\begin{aligned}\{[T_{6m-1}\cdots T_1][(T_{6m-1}T_{6m-2})^{-1}(T_{6m-4}T_{6m-5})^{-1}\\ \cdots(T_2T_1)^{-1}(T_{6m-1}T_{6m-2})^2(T_{6m-4}T_{6m-5})^2\\ \cdots(T_2T_1)^2(T_{6m-1}T_{6m-2})^{-1}(T_{6m-4}T_{6m-5})^{-1}\\ \cdots(T_2T_1)^{-1}][T_{6m-1}\cdots T_1]\}.\end{aligned}$$

This arrangement is possible due to the fact that the center bracket reduces to the identity in virtue of the relation (6). If we again make use of (6), this expression may be rearranged as follows:

$$\begin{aligned}\{[T_{6m-1}T_{6m-2}T_{6m-3}\cdot(T_{6m-1}T_{6m-2})^{-1}T_{6m-4}T_{6m-5}T_{6m-6}\cdot(T_{6m-4}T_{6m-5})^{-1}\\ \cdots T_5T_4T_3\cdot(T_5T_4)^{-1}T_2T_1(T_2T_1)^{-1}][(T_{6m-1}T_{6m-2})^2(T_{6m-4}T_{6m-5})^2\\ \cdots(T_2T_1)^2][(T_{6m-1}T_{6m-2})^{-1}T_{6m-1}T_{6m-2}(T_{6m-4}T_{6m-5})^{-1}\cdot T_{6m-3}T_{6m-4}T_{6m-5}\\ \cdots(T_2T_1)^{-1}\cdot T_3T_2T_1]\}.\end{aligned}$$

If we perform the obvious cancellations and apply Lemma 1 to the last bracket, we obtain:

$$\begin{aligned}\{[T_{6m-1}T_{6m-2}T_{6m-3}(T_{6m-1}T_{6m-2})^{-1}\cdots T_5T_4T_3(T_5T_4)^{-1}][(T_{6m-1}T_{6m-2})^2\\ \cdots(T_2T_1)^2]T_{6m-3}T_{6m-4}T_{6m-5}(T_{6m-3}T_{6m-4})^{-1}\cdots T_3T_2T_1(T_3T_2)^{-1}]\},\end{aligned}$$

and, applying Lemma 2 to the elements of the middle bracket, we are thus enabled to write $T'^2$ in the form

$$
\begin{aligned}
(14)\quad T'^2 = \{&[T_{6m-1}T_{6m-2}T_{6m-3}(T_{6m-1}T_{6m-2})^{-1}\cdots T_5T_4T_3(T_5T_4)^{-1}] \\
&[(T^{-1}_{6m-1}T_{6m-2}T_{6m-1})T^3{}_{6m-2}(T^{-1}_{6m-4}T_{6m-5}T_{6m-4})T^3{}_{6m-5}\cdots(T_2{}^{-1}T_1T_2)T_1{}^3] \\
&[T_{6m-3}T_{6m-4}T_{6m-5}(T_{6m-3}T_{6m-4})^{-1}\cdots T_3T_2T_1(T_3T_2)^{-1}]\}^{3m},
\end{aligned}
$$

which is the desired type of factorization.

**5. A function $f^*_{6m}$, of type (3), whose critical values correspond to induced motions given by the factors of (14).** Consider the curve

$$
(15)\qquad F_{6m}(x,y) \equiv (x^{3m}-1)^2 - (y^{2m}-1)^3 = 0.
$$

The critical values of the function $F_{6m}(x,y)$ are divided into two classes according to the following classification:

1. $x = a_j = \omega^j{}_{3m}$ $(j = 0, 1, \cdots, 3m-1)$, where $\omega^j{}_{3m}$ denote the $3m$-th roots of unity. Each of these critical values corresponds to a line containing $2m$ cusps of the curve $F_{6m} = 0$, each having a vertical cusp tangent.
2. $x = c_{k,l}, = |\,2\,|^{1/6m} \cdot e\left[i\pi \dfrac{(-1)^k + 8l}{12m}\right]$ for $k = 0, 1$ and $l = 0, 1, \cdots, 3m-1$. Each of these values corresponds to a flex tangent having contact of order $2m-1$ with the curve $F_{6m} = 0$.

Let us choose $x = 0$ as common origin of loops $\delta'_j$, $\delta'_{kl}$ in the $x$: plane selected as indicated in Fig. 4. The roots $y_k = b_k$ of $F_{6m}(0,y) = 0$ are given by:

$$
\begin{aligned}
b_{1+3j} &= e\left[i\pi \frac{(6j-1)}{6m}\right] \\
b_{2+3j} &= |\,2\,|^{1/2m} \cdot e^{i\pi j/m} \\
b_{3+3j} &= e\left[i\pi \frac{(6j+1)}{6m}\right]
\end{aligned}
$$

for $j = 0, 1, \cdots, 2m-1$. We now choose oriented arcs $\bar{\sigma}_k$ as indicated in the diagram of the $y$: plane of Fig. 4 and choose loops $g_k$, surrounding $b_k$, in the manner outlined in Sec. 4. Let us examine the motion induced on the roots $y_k$ as $x$ describes one of the loops $\delta'$ in the $x$: plane. This examination will be somewhat simplified if we also consider the auxiliary curve

$$
(16)\qquad Y^3 = X^2
$$

obtained from the curve $F_{6m}(x,y) = 0$ by setting $Y = y^{2m} - 1$ and $X = x^{3m} - 1$. The critical values of the function $F_{6m}(x,y)$ are evidently divided into three classes: the values $a_j$ corresponding to the cusp of (16), the values $c_{0,l}$ corresponding to the ordinary value $X = i$ and the values $c_{1,l}$ corresponding to the

ordinary value $X = -i$. The initial value, $x = 0$, of the loops $\delta'$ corresponds to the ordinary value $X = 1$. It is therefore clear that as $x$ describes a loop $\delta'_j$, $X$ describes a loop $\Delta$, and, as $x$ describes a loop $\delta'_{kl}$, $X$ describes a loop $\Delta_k$ where $\Delta$, $\Delta_0$, $\Delta_1$ are loops in the $X$: plane emanating from $X = 1$ and surrounding respectively the values $X = 0, i, -i$. If we denote the roots $Y_n$ of (16) for $X = 1$ by $B_n$ where $B_n = e[(2\pi i/3)(n-2)]$ $(n = 1, 2, 3)$, then, clearly, the image of the points $b_{n+3j}$ in the $Y$: plane is the point $B_n$ for all values of $j$. Let us define elementary motions $T_i$ of the points $b_k$ with reference

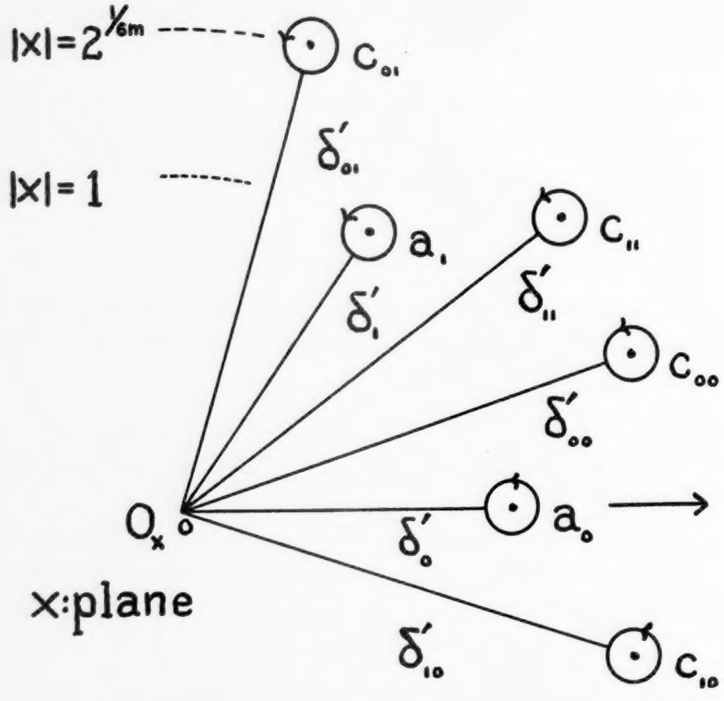



Fig. 4x.

to the oriented arcs $\bar{\sigma}_k$ of Fig. 4, and elementary motions $T^*_i$ of the points $B_k$ with reference to arcs $\bar{\Sigma}_i$ chosen as positively oriented arcs of the unit circle joining $B_i$ to $B_{i+1}$ for $i = 1, 2$. As $x$ describes the loop $\delta'_s$, $X$ describes the loop $\Delta$ and, therefore, the points $B_n$ undergo the motion $(T^*_2T^*_1)^2$. It therefore follows that the image points, $b_{n+3j}$, in the $y$: plane are subjected to the motion $(T_{2+3j}T_{1+3j})^2$ for all values of $j$. Thus, as $x$ describes the loop $\delta'_s$, we have that the corresponding motion of the points $b_k$ is given by

$$(17) \qquad (T_2T_1)^2(T_5T_4)^2 \cdots (T_{6m-1}T_{6m-2})^2$$

for $s = 0, 1, \cdots, 2m-1$.

Suppose $x$ describes the loop $\delta'_{ks}$; then $X$ describes the loop $\Delta_k$. Since $\Delta_k$ does not surround a critical value of the function (16), it follows that the motion $T^*$ of the points $B_n$ is the identity. Consequently, the motion of the points $b_k$ must leave the set $[b_{n+3j}]_{(j)}$ invariant for each $n$. This motion is due to the flex line $x = c_{k,s}$ of the pencil $\{x = \text{const.}\}$ and interchanges the points $b_{(3-2k)+3j}$ in a cyclical manner. This motion has the following description in terms of the elementary motions $T_i$:

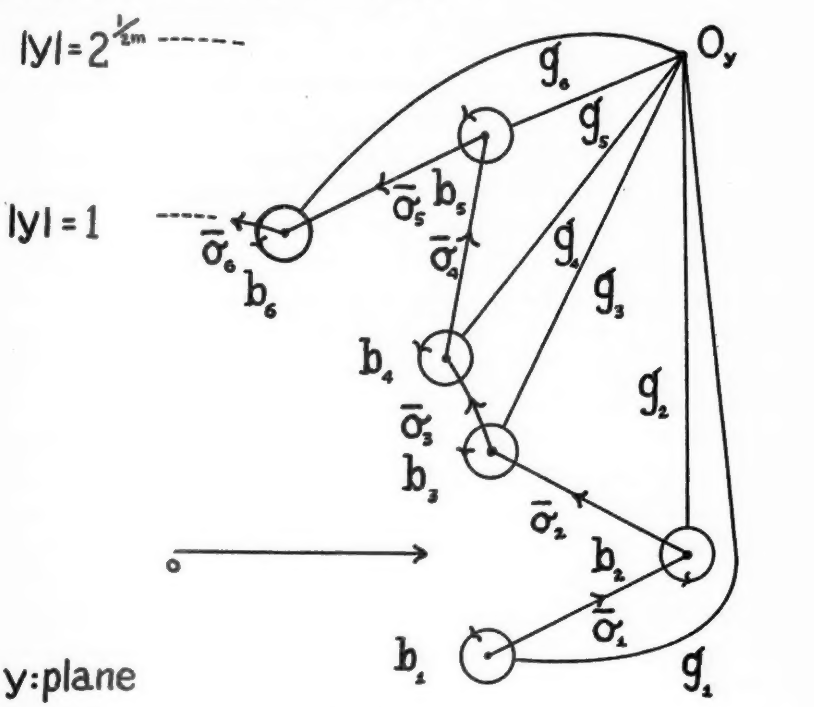



Fig. 4$y$.

$$(18)\qquad T^{(k)} = \boxed{\prod_{j=0}^{2m-2}}_{R} \left(T_{6m+2k-3j-3}T_{6m+2k-3j-4}T_{6m+2k-3j-5}\left(T_{6m+2k-3j-3}T_{6m+2k-3j-4}\right)^{-1}\right)$$

where $\boxed{\prod}_{R}$ indicates multiplication on the right.

For the curve $F_{6m}(x, y) = 0$, certain of the singular lines of the pencil $\{x = \text{const.}\}$ are coincident. These coincidences are not intrinsic and are due to the special choice of the coördinate axes. In order to eliminate these coincidences from the considerations, we proceed in the following manner.

If we denote by $F_{6m}(\theta)$ the function obtained by rotating $F_{6m}$ through a positive angle $\theta$, then, for small values of $\theta$, the curve $F_{6m}(\theta) = 0$ will have cusps near to the cusps of $F_{6m} = 0$; however, the cusp tangents will no longer

be vertical. In fact, the critical value $x = a_j$ of $F_{6m}$ has associated with it critical values $x = a_{jk}$ corrresponding to cusps of $F_{6m}(\theta) = 0$ and values $x = \bar{a}_{jk}$ corresponding to simple tangents of the curve $F_{6m}(\theta) = 0$, where $a_{jk}$ and $\bar{a}_{jk}$ are in the vicinity of $a_j$, for $k = 1, 2, \cdots, 2m$.

When a singular line of the pencil $\{x = \text{const.}\}$, which passes through a cusp having this singular line as cusp tangent, breaks up into a pair of singular lines, one of which is a simple tangent and the other a line through the cusp, the motion $(T_{s+1}T_s)^2$ breaks up into either the product of $T^3{}_{s+1}$ and $T^{-1}_{s+1}T_sT_{s+1}$ or of $T_s{}^3$ and $T^{-1}_{s+1}T_sT_{s+1}$, according as which of the two, essentially distinct, possible choices of loops in the $x$: plane is selected. If we are interested merely in the generating relations of the fundamental group and not in the actual motion of the roots, it is indifferent which of these choices is made, due to the fact that the resulting relations among the generators of the fundamental group are the same in both cases.

The singular lines of the pencil $\{x = \text{const.}\}$ passing through flexes of $F_{6m} = 0$ break up into distinct singular lines passing through points of simple tangency of $F_{6m}(\theta) = 0$. Once more, there is ambiguity concerning the determination of the actual motions corresponding to these singular lines, the motions again being dependent on the choice of loops. However, the flex as a unit imposes the same relations on the generators $g_k$ as does the corresponding number of distinct simple tangents which approach coincidence to form the flex. Consequently, for any choice of loops, the motions corresponding to these simple tangents must impose the same relations on the elements $g_k$ as the factors of the flex motion (18) which correspond to simple tangents. Thus, from the standpoint of relations on the generators $g_k$, it is sufficient to treat the set of singular lines which approach coincidence in a flex tangent as a unit and merely say that the imposed relations are those of the original flex, the individual motions corresponding to the tangents being left out of the consideration entirely.

To summarize, $F_{6m}(\theta)$ will have the following properties:

1°. Possesses $6m^2$ critical values $a_{jk}$ corresponding to cusps of the curve $F_{6m}(\theta) = 0$ such that, for a proper choice of loops in the $x$: plane surrounding these values, the induced motions are $(T_{3k+1})^3$ where $k = 0, 1, \cdots, 2m-1$ and $j = 0, 1, \cdots, 3m-1$.

2°. Possesses $6m^2$ critical values $\bar{a}_{jk}$ corresponding to simple tangents of the curve $F_{6m}(\theta) = 0$, such that, for a proper choice of loops in the $x$: plane, the induced motions are $T^{-1}_{3k+2}T_{3k+1}T_{3k+2}$ for $k = 0, 1, \cdots, 2m-1$ and $j = 0, 1, \cdots, 3m-1$.

3°. Possesses $6m$ sets of $2m-1$ critical values $c_{klt}$ corresponding to

simple tangents of the curve $F_{6m}(\theta) = 0$ of such a type that, for a proper choice of loops in the $x$: plane, the induced motions relative to a set $c_{klt}$ for $k$ and $l$ fixed impose the same relations on the generators $g_j$ as do the motions

$$(19) \qquad T_{6m+2k-3t-3} T_{6m+2k-3t-4} T_{6m+2k-3t-5} (T_{6m+2k-3t-3} T_{6m+2k-3t-4})^{-1}$$

for $t = 0, 1, \cdots, 2m - 2$, where $k$ and $l$ may assume any of the values $k = 0, 1$; $l = 0, 1, \cdots, 3m - 1$.

These properties are exactly those desired for the function $f^*_{6m}$ and, accordingly, we define $f^*_{6m}$ to be the function $F_{6m}(\theta)$.

**6. Determination of the structure of the fundamental group $G^*$ of the curve $f^*_{6m}(x, y) = 0$.** The component transformations of $t'^2$ associated with the motions corresponding to the critical values of the function $f^*_{6m}(x, y)$ give rise to relations among the generators $g_k$ of $G^*$ of the following types:

1°. From the transformations associated with the motions (19), we obtain the relations

$$(20.j) \qquad g_{6m-3j-2} g_{6m-3j-1} g_{6m-3j} = g_{6m-3j-1} g_{6m-3j} g_{6m-3j+1}$$

and the relations

$$(21.j) \qquad g_{6m-3j} g_{6m-3j+1} g_{6m-3j+2} = g_{6m-3j+1} g_{6m-3j+2} g_{6m-3j+3}$$

where $j = 1, 2, \cdots, 2m - 1$.

2°. The transformations $t^3_{6m-3j+1}$ give the relations

$$(22.j) \qquad g_{6m-3j+1} g_{6m-3j+2} g_{6m-3j+1} = g_{6m-3j+2} g_{6m-3j+1} g_{6m-3j+2}$$

for $j = 1, 2, \cdots, 2m$.

3°. The transformations $t^{-1}_{6m-3j+2} t_{6m-3j+1} t_{6m-3j+2}$ yield the relations

$$(23.j) \qquad g_{6m-3j+1} = g_{6m-3j+3} \text{ for } j = 1, 2, \cdots, 2m).$$

In addition, the generators $g_k$ also satisfy the trivial relation

$$(4) \qquad g_1 g_2 \cdots g_n = 1.$$

If we now apply $(23.j+1)$ to $(20.j)$ we have

$$g_{6m-3j-2} g_{6m-3j-1} g_{6m-3j-2} = g_{6m-3j-1} g_{6m-3j-2} g_{6m-3j+1}$$

and, on application of $(22.j+1)$ to the left-hand member of this expression, we obtain

$$g_{6m-3j-1} g_{6m-3j-2} g_{6m-3j-1} = g_{6m-3j-1} g_{6m-3j-2} g_{6m-3j+1}$$

whence

$$(24) \qquad g_{6m-3j-1} = g_{6m-3j+1} \text{ for } j = 1, \cdots, 2m - 1.$$

In the same way, if we apply $(23.j)$ to $(21.j)$ we obtain

$$g_{6m-3j} g_{6m-3j+1} g_{6m-3j+2} = g_{6m-3j+1} g_{6m-3j+2} g_{6m-3j+1}$$

and, by application of $(22.j)$ to the right-hand member, this expression reduces to

$$g_{6m-3j}g_{6m-3j+1}g_{6m-3j+2} = g_{6m-3j+2}g_{6m-3j+1}g_{6m-3j+2}$$

whence,

$$g_{6m-3j} = g_{6m-3j+2} \text{ for } j = 1, \cdots, 2m-1. \tag{25}$$

On combining the relations (23), (24) and (25) we obtain

$$g_1 = g_3 = \cdots = g_{6m-1}; \qquad g_2 = g_4 = \cdots = g_{6m}. \tag{26}$$

If we make use of the relations (26), we are enabled to eliminate the generators $g_3, g_4, \cdots, g_{6m}$ from the relation (4) obtaining

$$(g_1g_2)^{3m} = 1. \tag{27}$$

Thus, $G^*$ may be generated by two elements $g_1$ and $g_2$ satisfying the relations $(22.2m)$ and (27). The relation $(22.2m)$ gives as a consequence the relation $(g_1g_2)^3 = (g_1g_2g_1)^2$ and conversely.

Let us define new elements $U$ and $V$ in the following manner:

$$U = g_1g_2g_1; \qquad V = g_1g_2. \tag{28}$$

Then the relations:

$$U^2 = V^3; \qquad U^{2m} = 1, \tag{29}$$

together with the defining relations (28), give as consequences the relations (27) and $(22.2m)$. Hence, it is possible to generate the fundamental group, $G^*$, of the curve $f^*_{6m}(x,y) = 0$ by the two elements $U$ and $V$ satisfying the relations (29).

**7. Conclusion.** The curve $f^*_{6m} = 0$ is of the same type as $f_{6m} = 0$. Moreover, the number and kind of singularities are the same for both curves and the restrictive hypothesis to the effect that the cusps of $f_{6m}(x,y) = 0$ should be distinct is also satisfied for $f^*_{6m}(x,y) = 0$. Consequently, it is clear that the curves are isotopic and, therefore, that their fundamental groups possess the same structure. Hence, we may conclude that the fundamental group $G$ of the curves

$$f_{6m}(x,y) \equiv [\phi_{3m}(x,y)]^2 + [\psi_{2m}(x,y)]^3 = 0 \tag{3}$$

may be generated by two elements $U$ and $V$, of respective orders $2m$ and $3m$, satisfying the relations $U^2V^{-3} = U^{2m} = 1$.

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# GEOMETRY OF TURBINES, FLAT FIELDS, AND DIFFERENTIAL EQUATIONS.*

By Edward Kasner and John De Cicco.

In this paper, we study the geometry of the oriented lineal elements of a plane. We give additional results to those found in a paper by the senior writer entitled "The group of turns and slides and the geometry of turbines," published in 1911 in the *American Journal of Mathematics,* vol. 33, pp. 193-202. The present paper, however, can be read independently of the earlier paper.

We define $\infty^1$ elements to be a *series* of elements; this includes a union or curve as a special case. Of course, $\infty^2$ elements form a *field* of elements, which corresponds to a differential equation of first order $F(x, y, y') = 0$. A *turbine* is the series, which is obtained by converting each element of an oriented circle into one having the same point and a direction making a fixed angle $\alpha$ with the original direction. A *flat field* is the field that is obtained from the totality of all elements which are determined by either the set of all oriented circles containing a given element, or as a special case, the set of all oriented lines, which are parallel to and possess the same orientation as a given line. We desire to study the relationships between general series, general fields (differential equations), turbines and flat fields.

For the analytic representation of an element, it will be found convenient to use two systems of coördinates, called the cartesian and hessian coördinate systems respectively. The cartesian coördinates of an element $E$ are either $(x, y, y')$ or else $(x, y, \theta)$, where $(x, y)$ are the cartesian coördinates of the point of the element and $\theta$ is the inclination of the line of the element. The hessian coördinates of an element are $(u, v, w)$ where $v$ is the perpendicular from the origin to the line of the element $E$, $u$ is the angle between the perpendicular and the initial line, and $w$ is the distance between the foot of the perpendicular and the point of the element.

The final theorems constitute wide extensions of Scheffer's[1] theory of isogonal trajectories and equi-tangential trajectories, including his two dual theorems as very special cases.

The *main theorems* in our paper are those numbered 8, 14, 15, 16, 19,

* Received November 13, 1936; Revised December 18, 1936.

[1] *Mathematische Annalen,* vol. 60 (1905), pp. 491-531.

20, 23, 30, 33, 35, 36. The theory of *conjugate differential equations* (possessing the same $\infty^2$ osculating circles, Theorem 19) is noteworthy.

**The turbine.** A turbine is the set of oriented elements which are obtained by converting each oriented element of an oriented circle into one having the same point and a direction making a fixed angle $\alpha$ with the original direction. We call the turbine non-linear or linear according as the circle is non-linear or linear.

In cartesian coördinates, the equations of a non-linear turbine are

$$x = a + R \sin(\theta + \alpha), \qquad y = b - R\cos(\theta + \alpha);$$

and in hessian coördinates, the equations are

$$v = a \cos u + b \sin u + r, \qquad w = -a \sin u + b \cos u + s,$$

where

$$r = R \cos \alpha, \qquad s = R \sin \alpha.$$

These equations show that the points of the elements of a turbine form a circle, which we call the *outer circle* of the turbine; and that the lines of the elements are the tangent lines of a circle, which we call the *inner circle.*

In cartesian coördinates, the equations of a linear turbine are

$$x \cos u + y \sin u = v, \qquad \theta + \alpha = u + \pi/2 + 2k\pi,$$

where $u$ and $v$ are constants and, in hessian coördinates, the equations of a linear turbine are

$$U = u - \alpha + 2k\pi, \qquad V \cos \alpha + W \sin \alpha = v.$$

We obtain the following two theorems:

THEOREM 1. *Two elements determine a unique turbine.*

THEOREM 2. *Two turbines possess either no common elements or one common element.*

If a number of elements are all on a turbine, we shall say that these elements are *co-turbinal.*

If a number of turbines all have one element in common, we shall say that these turbines are *co-elemental.*

Let $T$ be the turbine, which is obtained by converting each element of the oriented circle $C$ into one having the same point and a direction making a fixed angle $\alpha$ with the original direction. Then the turbine $NT$ is defined to be the *conjugate turbine* of $T$, if it is obtained by converting each element of the

oriented circle $C$ into one having the same point and a direction making a fixed angle $-\alpha$ with the original direction. (By means of a certain representation $R$ of the elements of the plane by the points of space, studied in the paper of 1911, conjugacy is defined as polarity with respect to a basic null-system $N$).

In hessian coördinates, the conjugate turbine $NT$ of the non-linear turbine $T$

$$v = a \cos u + b \sin u + r, \qquad w = -a \sin u + b \cos u + s,$$

is

$$v = a \cos u + b \sin u + r, \qquad w = -a \sin u + b \cos u - s.$$

The conjugate turbine $NT$ of the linear turbine $T$

$$U = u - \alpha + 2k\pi, \qquad V \cos \alpha + W \sin \alpha = v,$$

is

$$U = u + \alpha + 2l\pi, \qquad V \cos \alpha - W \sin \alpha = v.$$

Theorem 3. *The conjugate turbines of two given turbines possess no common elements or one common element according as the two given turbines possess no common elements or one common element.*

**The flat field.** The totality of elements determined by the set of all oriented circles, which contain a given element, is called a non-linear flat field. The given element is called the center or the central element of the flat field. *All the elements of the field are thus co-circular with a fixed (central) element.*

In cartesian coördinates, the non-linear flat field is given by

$$\theta = -\gamma - 2 \operatorname{arc} \tan \frac{x - \alpha}{y - \beta} + 2k\pi,$$

where $(\alpha, \beta, \gamma)$ are the cartesian coördinates of the element which is the negative (or reverse) of the given element contained in the oriented circles. In hessian coördinates the non-linear flat field is given by

$$w = (v + b) \tan \tfrac{1}{2}(a - u) + c,$$

where $(a, b, c)$ are the hessian coördinates of the element, which is the negative of the given element.

The set of all elements obtained by setting $u =$ constant, say $\alpha$, is called a linear flat field.

It is easy to prove the following theorems:

Theorem 4. *Three elements determine a unique flat field.*

Theorem 5. *Two flat fields have in common one and only one turbine.*

Theorem 6. *Three flat fields have in common one and only one element, or else they have a turbine in common.*

**The envelopes of a one parameter family of series of elements.** We define $\infty^1$ elements to be a series of elements. The points of the elements of a series form a curve which we call the point-curve of the series, and the lines of the elements of a series are the tangent lines of a curve which we call the line curve of the series.

Now we consider a one parameter family of series of elements. Let us determine the envelope of the one parameter family of point-curves and the envelope of the one parameter family of line-curves of the one parameter family of series of elements.

Now consider any particular series of the one parameter family of series. The point of intersection of the envelope of the one parameter family of point-curves and of the point-curve of this particular series belongs to an element of this particular series. This element is defined to be an element of the point-envelope of the one parameter family of series of elements.

Again consider any particular series of the one parameter family of series. The common tangent line of the envelope of the one parameter family of line-curves and of the line-curve of this particular series contains an element of this particular series. This element is defined to be an element of the line envelope of the one parameter family of series of elements.

If the one parameter family of series is given in cartesian coördinates by the equations

$$y = f(x, t), \qquad \theta = g(x, t),$$

where $t$ is the parameter, then the point-envelope is given by the equations

$$y = f(x, t), \qquad \theta = g(x, t), \qquad f_t(x, t) = 0.$$

If the one parameter family of series is given in hessian coördinates by the equations

$$v = F(u, t), \qquad w = G(u, t),$$

where $t$ is the parameter, then the line envelope is given by the equations

$$v = F(u, t), \qquad w = G(u, t), \qquad F_t(u, t) = 0.$$

THEOREM 7. *For the point envelope and the line envelope of a one parameter family of series of elements to be identical, it is necessary and sufficient that either the one parameter family of series be a one parameter family of oriented curves; or, if the one parameter family of series is given in cartesian coördinates by the equations*

$$x = \phi(\theta, t), \qquad y = \psi(\theta, t),$$

*where t is the parameter, the eliminant with respect to $\theta$ of the two equations*

$$\phi_t(\theta, t) = 0, \qquad \psi_t(\theta, t) = 0,$$

*be identically zero, or if the one parameter family of series is given in hessian coördinates by the equations*

$$v = f(u, t), \qquad w = g(u, t),$$

*where t is the parameter, the eliminant with respect to u of the two equations*

$$f_t(u, t) = 0, \qquad g_t(u, t) = 0,$$

*be identically zero.*

If a family of series of elements is given in cartesian coördinates by the equations

$$x = \phi(\theta, t), \qquad y = \psi(\theta, t),$$

and, if the eliminant with respect to $\theta$ of the two equations

$$\phi_t(\theta, t) = 0, \qquad \psi_t(\theta, t) = 0,$$

is identically zero, we define the series, which is obtained from the equations

$$x = \phi(\theta, t), \qquad y = \psi(\theta, t),$$

where $\theta = \theta(t)$ is the common solution of the two equations

$$\phi_t(\theta, t) = 0, \qquad \psi_t(\theta, t) = 0,$$

to be the envelope of the family of series of elements.

If a one parameter family of series of elements is given in hessian coördinates by the equations

$$v = f(u, t), \qquad w = g(u, t),$$

and if the eliminant with respect to $u$ of the two equations

$$f_t(u, t) = 0, \qquad g_t(u, t) = 0,$$

is identically zero, we define the series, which is obtained from the equations

$$v = f(u, t), \qquad w = g(u, t),$$

where $u = u(t)$ is the common solution of the two equations

$$f_t(u, t) = 0, \qquad g_t(u, t) = 0,$$

to be the envelope of the one parameter family of series of elements.

It is easy to prove that the above two definitions are equivalent.

THEOREM 8. *The necessary and sufficient condition that the one parameter family of turbines*

$$v = a(t)\cos u + b(t)\sin u + r(t), \qquad w = -a(t)\sin u + b(t)\cos u + s(t)$$

*possess an envelope is that*

$$a'^2 + b'^2 = r'^2 + s'^2.$$

Moreover the envelope is unique and it is given by the equations

$$\cos u = \frac{-a'r' - b's'}{a'^2 + b'^2}, \qquad \sin u = \frac{a's' - b'r'}{a'^2 + b'^2},$$
$$v = a\cos u + b\sin u + r, \qquad w = -a\sin u + b\cos u + s.$$

For the eliminant of the equations

$$a'\cos u + b'\sin u + r' = 0, \qquad -a'\sin u + b'\cos u + s' = 0,$$

is obviously the above condition. The series is unique since $\cos u$, $\sin u$, $v$, $w$ satisfy linear equations. The theorem follows.

From Theorem 8 we obtain the following:

THEOREM 9. *The one parameter family of conjugate turbines does or does not possess an envelope according as the given one parameter family of turbines does or does not possess an envelope.*

THEOREM 10. *The necessary and sufficient conditions that the one parameter family of turbines*

$$v = a(t)\cos u + b(t)\sin u + r(t), \qquad w = -a(t)\sin u + b(t)\cos u + s(t),$$

*be all co-elemental are*

$$a'^2 + b'^2 = r'^2 + s'^2, \qquad a'b'' - a''b' = r's'' - r''s'.$$

**The tangent turbines of a series of elements.** Any series of elements, which has the property that consecutive elements are non-parallel, may be given in hessian coördinates by the equations

$$v = v(u), \qquad w = w(u).$$

In what follows, we use hessian coördinates.

Let $u$ and $u + \Delta u$ determine the two elements of the series

$$(u, v, w) \quad \text{and} \quad (u + \Delta u, v(u + \Delta u) = v + \Delta v, w(u + \Delta u) = w + \Delta w).$$

Since these two elements cannot be parallel, they determine a unique non-linear turbine, which is given by the parameter values

$$a = \frac{1}{2 \sin \frac{1}{2}\Delta u} [-\Delta v \sin \tfrac{1}{2}(2u + \Delta u) - \Delta w \cos \tfrac{1}{2}(2u + \Delta u)],$$
$$b = \frac{1}{2 \sin \frac{1}{2}\Delta u} [\Delta v \cos \tfrac{1}{2}(2u + \Delta u) - \Delta w \sin \tfrac{1}{2}(2u + \Delta u)],$$
$$r = v + \tfrac{1}{2}\Delta v + \tfrac{1}{2}\Delta w \cot \tfrac{1}{2}\Delta u,$$
$$s = w + \tfrac{1}{2}\Delta w - \tfrac{1}{2}\Delta v \cot \tfrac{1}{2}\Delta u.$$

The limiting turbine (of the above set of turbines), as $\Delta u$ approaches zero, is given by the parameter values

$$a = -v'(u) \sin u - w'(u) \cos u,$$
$$b = v'(u) \cos u - w'(u) \sin u,$$
$$r = v(u) + w'(u),$$
$$s = -v'(u) + w(u).$$

We call this turbine the *tangent turbine* of the series at the element $(u, v, w)$.[2]

It is found that the rate of turning with respect to the arc length of the point-curve of the series of the element of the series is $\pm 1/R$ where $R$ is the radius of the outer circle.

THEOREM 11. *The necessary and sufficient conditions that a one parameter family of turbines be a set of tangent turbines of a series of elements, are that the one parameter family of turbines be not a co-elemental family of turbines and possess an envelope. Moreover the envelope is the series to which the turbines are the tangent turbines.*

The proof of Theorem 11 follows immediately from a consideration of the equations for the parameter values of the tangent turbines of a series.

From Theorem 7 and Theorem 11 we obtain

THEOREM 12. *The necessary and sufficient conditions, that a single infinitude of turbines be a set of tangent turbines to a series, are* (1) *that the set of turbines be not co-elemental and,* (2) *if the turbines are not all circles, the line and point-envelopes of the turbines be coincident and, if the turbines are all circles, the envelope of the circles consist of one curve.*

THEOREM 13. *If a one parameter family of turbines is a set of tangent turbines, then the conjugate turbines are either a set of tangent turbines or a set of co-elemental turbines.*

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[2] If the series is a curve (union), the tangent turbine at the element $E$ becomes the osculating circle of the curve.

**The osculating flat fields of series of elements.** The flat field, which has three consecutive elements in common with a series at an element $E$ of the series, is called the osculating flat field of the series at $E$.

Let the flat field

$$w = (v + b) \tan \tfrac{1}{2}(a - u) + c,$$

be the osculating flat field of the series

$$v = v(u), \qquad w = w(u),$$

at the element $E$. Then if $\alpha$, $\beta$, $r$, $s$ are the parameter values of the tangent turbine of the series at $E$, we must have

$$\cos a = \frac{\alpha' r' - \beta' s'}{\alpha'^2 + \beta'^2}, \qquad \sin a = \frac{\alpha' s' + \beta' r'}{\alpha'^2 + \beta'^2},$$
$$b = \alpha \cos a + \beta \sin a - r, \qquad c = -\alpha \sin a + \beta \cos a + s.$$

These equations show that the centers (central elements) of the osculating flat fields of the series are the elements of the envelope of the conjugate turbines.

**General fields of elements.** The two parameter family of elements, which is given in cartesian coördinates by the equation

$$\theta = g(x, y),$$

or in hessian coördinates by the equation

$$w = f(u, v),$$

where $f$ is of period $2\pi$ in $u$, is called a field of elements.

Let the field be given in hessian coördinates by $w = f(u, v)$. Then, if $v = v(u)$, the functions $v = v(u)$, $w = f(u, v(u))$ define a series of elements, which we call a field series of the field.

Now let a field series contain the element $(u, v, w = f(u, v))$. The parameter values of the tangent turbine of this field series at the element $(u, v, w = f(u, v))$ are given by

$$\begin{aligned} a &= -v'(\sin u + f_v \cos u) - f_u \cos u, \\ b &= v'(\cos u - f_v \sin u) - f_u \sin u, \\ r &= v + f_u + v' f_v, \\ s &= f - v'. \end{aligned}$$

If the field is given in cartesian coördinates by $\theta = g(x, y)$, then the parameter values of the tangent turbine of the field series at the element $(x, y, \theta = g(x, y))$ are

$$a = x - y'/(g_x + y'g_y),$$
$$b = y + 1/(g_x + y'g_y),$$
$$R = \pm\sqrt{1+y'^2}/(g_x + y'g_y),$$
$$\cos(\theta+\alpha) = \pm\frac{1}{\sqrt{1+y'^2}}, \qquad \sin(\theta+\alpha) = \pm\frac{y'}{\sqrt{1+y'^2}}.$$

From these formulae, we easily obtain the following ***fundamental theorems*** *on the structure of a field* in the neighborhood of any one of its elements.

THEOREM 14. *Consider any field $F$ and any element $E$ of the field. Then we study the totality of series starting at $E$ and contained in the field. From this totality, select that subset of series whose point-loci pass through the point of $E$ in a given direction and whose line loci thereby necessarily touch the line of $E$ at a fixed point. This subset (although it contains $\infty^\infty$ series) determines a unique tangent turbine.*

THEOREM 15. *By varying the given direction in Theorem 14 we thus obtain $\infty^1$ turbines. These turbines have their centers on a straight line. This line we shall call the central line relative to the given field $F$ and the given element $E$.*

From Theorem 15 we obtain

THEOREM 16. *The outer circles of the $\infty^1$ turbines of Theorem 15 form a pencil in the sense of elementary circle geometry, that is, the circles have two points in common. One of the fixed points is, of course, the point of the element $E$ and the other is a new point which we denote by $\bar{P}$. The inner circles of the turbines form a pencil in the sense of higher circle geometry, that is, the circles have two lines in common. One of the fixed lines is of course the line of the element $E$ and the other is a new line which we denote by $\bar{l}$.*

The central line is given either by the equation

$$x(\cos u - f_v \sin u) + y(\sin u + f_v \cos u) + f_u = 0,$$

or by the equation

$$-g_y(X-x) + g_x(Y-y) = 1.$$

The hessian coördinates of this straight line are determined by

$$\cos(U-u) = \frac{1}{\pm\sqrt{1+f_v^2}}, \qquad \sin(U-u) = \frac{f_v}{\pm\sqrt{1+f_v^2}},$$
$$V = \frac{f_u}{\mp\sqrt{1+f_v^2}}.$$

**The tangent flat field.** We say that the two fields $f$ and $F$ are tangent to each other at a common element $E$, if any two field series of the fields $f$ and $F$ respectively, which contain the element $E$ and which have the property that either their line curves or their point curves have a common tangent element at $E$, are such that their tangent turbines at $E$ are identical.

It is then obvious that two fields $w = f(u, v)$ and $w = F(u, v)$ are tangent at a common element $E$ if

$$f_u = F_u, \qquad f_v = F_v.$$

Similarly the two fields $\theta = g(x, y)$ and $\theta = G(x, y)$ are tangent at a common element $E$, if

$$g_x = G_x, \qquad g_y = G_y.$$

We call the flat field, which is tangent to a field $F$ at an element $E$, the tangent flat field of the field $F$ at $E$.

Let the flat field

$$w = (v + b) \tan \tfrac{1}{2}(a - u) + c,$$

be the tangent flat field of the field

$$w = f(u, v),$$

at the element $E(u, v, w = f(u, v))$. Then we must have

$$\begin{aligned} a &= u + 2 \operatorname{arc} \tan f_v + 2k\pi, \\ b &= -v - 2f_u/(1 + f_v^2), \\ c &= f(u, v) + 2f_u f_v/(1 + f_v^2). \end{aligned}$$

**One parameter family of fields envelope—Characteristics.** The equation

$$w = f(u, v, a),$$

defines a one parameter family of fields.

The series, which is a field series of each of two consecutive fields of the family, is called a characteristic of the field. The locus of all the characteristics of the one parameter family of fields is a field and we call it the envelope of the one parameter family of fields. The equations

$$w = f(u, v, a), \qquad f_a(u, v, a) = 0,$$

for each $a$ represent a characteristic of the family and, when we eliminate $a$ from the above two equations, the resulting equation represents the envelope of the family of fields.

It is easily seen that the envelope is tangent to each member of the family of fields at all elements of its characteristics.

The locus of the elements, which are common to consecutive characteristics of a one parameter family of fields, is called the edge of regression. The eliminants, with respect to $a$, of the equations

$$w = f(u, v, a), \qquad f_a(u, v, a) = 0, \qquad f_{aa}(u, v, a) = 0,$$

give the equations of the edge of regression.

It is obvious that the tangent turbines of the edge of regression and any characteristic at a common element are identical.

**Developable fields.** The envelope of a one parameter family of flat fields is called a developable field. The characteristics of the one parameter family of flat fields are turbines and these turbines are called the generators of the developable field.

Since each flat field is tangent to the envelope along its characteristic, it follows that the tangent flat field to a developable field is the same at all elements of a generator. The edge of regression of the developable is the series to which the generators are the tangent turbines. Moreover, since consecutive generators are consecutive tangent turbines of the edge of regression, the osculating flat field of the series is that flat field of the family which contains these generators. But this flat field is tangent to the developable. Hence, the osculating flat field at any element of the edge of regression is the tangent flat field to the developable field.

THEOREM 17. *For the field $w = f(u, v)$ to be a developable field, it is necessary and sufficient that*

$$(1 + f_v^2 + 2f_{uv})^2 - 4f_{vv}(f_{uu} + f_u f_v) = 0.$$

For the tangent flat field at the element $(u, v, f(u, v))$ has parameter values

$$a = u + 2 \operatorname{arc} \tan f_v + 2k\pi, \qquad b = -v - 2f_u/(1 + f_v^2),$$
$$c = f + 2f_u f_v/(1 + f_v^2).$$

The necessary and sufficient condition that $b = b(a)$, $c = c(a)$ is then seen to be the above equation.

**Conjugate fields of elements.** We define the tangent turbines to any field series of a field to be the tangent turbines of the field. In hessian coördinates the tangent turbines of a field are given by the parameter values

$$a = -v'(\sin u + f_v \cos u) - f_u \cos u,$$
$$b = v'(\cos u - f_v \sin u) - f_u \sin u,$$
$$r = v + f_u + v' f_v,$$
$$s = f - v'.$$

The conjugate turbines of the above turbines are given by the parameter values

$$\bar{a} = a = -v'(\sin u + f_v \cos u) - f_u \cos u,$$
$$\bar{b} = b = v'(\cos u - f_v \sin u) - f_u \sin u,$$
$$\bar{r} = r = v + f_u + v' f_v,$$
$$\bar{s} = -s = -f + v'.$$

Since the tangent turbines of the field $w = f(u, v)$ at the element $(u, v, f(u, v))$ all contain the element $(u, v, f(u, v))$, the conjugate turbines of these turbines must also contain an element and it is unique. We call the element $(\bar{u}, \bar{v}, \bar{w})$ the conjugate of the element $(u, v, w)$. If $E$ is any element of the field $w = f(u, v)$, then we denote the conjugate element by $\bar{E}$. The element $\bar{E}$ is given in hessian coördinates by the equations

$$\cos(\bar{u} - u) = -\frac{1 - f_v^2}{1 + f_v^2}, \qquad \sin(\bar{u} - u) = -\frac{2f_v}{1 + f_v^2},$$
$$\bar{v} = v + 2f_u/(1 + f_v^2),$$
$$\bar{w} = -f(u, v) - 2f_u f_v/(1 + f_v^2).$$

and in cartesian coördinates, the element $\bar{E}$ is given by the equations,

$$\bar{x} = x - 2g_y/(g_x^2 + g_y^2),$$
$$\bar{y} = y + 2g_x/(g_x^2 + g_y^2),$$
$$\bar{\theta} = -g(x, y) + 2 \arctan g_y/g_x + (2k + 1)\pi.$$

From these equations, we obtain

THEOREM 18. *The necessary and sufficient condition, that the conjugate elements of a field be the elements of a field, is that the given field be non-developable.*

For, it is obvious that the necessary and sufficient condition, that the set of conjugate elements be at most a one parameter family of elements, is

$$(1 + f_v^2 + 2f_{uv})^2 - 4f_{vv}(f_{uu} + f_u f_v) = 0,$$

which means that the field $w = f(u, v)$ must be a developable field. The theorem follows.

If a field is non-developable, we term the field of conjugate elements the *conjugate field,* a fundamental concept in our theory.

From this follows

THEOREM 19. *Each tangent turbine of the conjugate field is the conjugate turbine of each tangent turbine of the given field. From this it follows that two conjugate families of curves have the same osculating circles.*

For a non-developable field, the equations:

$$\cos(\bar{u}-u) = -\frac{1-f_v^2}{1+f_v^2}, \qquad \sin(\bar{u}-u) = -\frac{2f_v}{1+f_v^2},$$
$$\bar{V} = v + 2f_u/(1+f_v^2),$$

define a line transformation. We call it the conjugate line transformation for the field.

For a non-developable field, the equations

$$X = x - 2g_y/(g_x^2 + g_y^2),$$
$$Y = y + 2g_x/(g_x^2 + g_y^2),$$

define a point transformation. We call it the conjugate point transformation for the field. The following four results are deduced:

THEOREM 20. *For a line transformation to be a conjugate line transformation of a field, it is necessary and sufficient that the corresponding $\bar{E}$ on $\bar{l}$ of any element $E$ on $l$ be in projective involution with the element $E'$ on $\bar{l}$, which is the tangent element on $\bar{l}$ of the oriented circle which contains the element $E$ and which is tangent to the line $\bar{l}$.*

THEOREM 21. *Let a line transformation be a conjugate line transformation. Then it is the conjugate line transformation of a unique field $w = \phi(u, v)$, which contains a given element $(u_0, v_0, w_0)$. Moreover, any other field, of which it is the conjugate line transformation, is obtained by applying a slide to the elements of the field $w = \phi(u, v)$.*

THEOREM 22. *For a point transformation to be the conjugate point transformation of a field, it is necessary and sufficient that the correspondent $\bar{E}$ on $\bar{P}$ of the element $E$ on $P$ be in projective involution with the element $E'$ on $\bar{P}$ which is the tangent element on $\bar{P}$ of the circle which contains the element $E$ and the point $\bar{P}$.*

THEOREM 23. *Let a point transformation be a conjugate point transformation. Then it is the conjugate point transformation of a unique field $\theta = \psi(x, y)$ which contains a given element $(x_0, y_0, \theta_0)$. Moreover, any other field, of which it is the conjugate point transformation, is obtained by applying a turn to the elements of the field $\theta = \psi(x, y)$.*

**The tangent turbines of a field.** Let us consider the tangent turbines of the field. The parameter values of the turbines are

$$\begin{aligned} a &= -w(\sin u + f_v \cos u) - f_u \cos u, \\ b &= w(\cos u - f_v \sin u) - f_u \sin u, \\ r &= v + f_u + w f_v, \\ s &= f - w, \end{aligned}$$

where the turbine determined by $u, v, w$ is the tangent turbine of any field series which contains the element $(u, v, f(u, v))$ and whose line curve contains the tangent element $(u, v, w)$ at the element $(u, v, f(u, v))$.

By means of the above equations we are able to prove the following theorems:

THEOREM 24. *For the set of tangent turbines of a field to be a three parameter family of turbines, it is necessary and sufficient that the field be not a flat field.*

THEOREM 25. *For the set of tangent turbines of a field to be a two parameter family of turbines, it is necessary and sufficient that the field be a flat field.*

THEOREM 26. *For a two parameter family of turbines to be the tangent turbines of a flat field, it is necessary and sufficient that the conjugate turbines all contain a given element. Moreover, the given element is the center of the flat field.*

THEOREM 27. *The necessary and sufficient condition, that every one parameter family of turbines of the tangent turbines of a field possess an envelope, is that the field be a flat field.*

THEOREM 28. *For the tangent turbines of a field to be field series of the field, it is necessary and sufficient that the field be a flat field.*

Let us now consider the three parameter family of turbines

$$\begin{aligned} v &= a(\lambda, \mu, \nu)\cos u + b(\lambda, \mu, \nu)\sin u + r(\lambda, \mu, \nu), \\ w &= -a(\lambda, \mu, \nu)\sin u + b(\lambda, \mu, \nu)\cos u + s(\lambda, \mu, \nu). \end{aligned}$$

Since the above set of turbines is a three parameter family of turbines, at least two of the jacobians

$$\frac{D(a, b, r)}{D(\lambda, \mu, \nu)}, \quad \frac{D(a, b, s)}{D(\lambda, \mu, \nu)}, \quad \frac{D(a, r, s)}{D(\lambda, \mu, \nu)}, \quad \frac{D(b, r, s)}{D(\lambda, \mu, \nu)},$$

are not identically zero.

A three parameter family of turbines, whose inner circles are all distinct, is called a general three parameter family of turbines. It is seen that the necessary and sufficient condition for a three parameter family of turbines to be a general set of turbines is that the jacobian

$$\frac{D(a, b, r)}{D(\lambda, \mu, \nu)}$$

be not identically zero. Hence, any general three parameter family turbines may be given by the equations

$$v = a \cos u + b \sin u + r, \qquad w = -a \sin u + b \cos u + s(a, b, r).$$

A three parameter family of turbines, such that it consists of turbines which contain an element of a fixed series, is called a co-serial set of turbines.

THEOREM 29. *The necessary and sufficient conditions for a three parameter family of turbines to be a co-serial set of turbines are that the family be a general set of turbines and that the equations*

$$s_a^2 + s_b^2 = 1 + s_r^2,$$
$$(1 + s_r^2)(s_{aa} + s_{bb}) + s_{rr} = (2s_a + s_b s_r)s_{br} + (-2s_b + s_a s_r)s_{ar},$$

*be identically satisfied.*

The fixed series is uniquely determined and is given by the equation

$$\cos u = \frac{-s_b + s_a s_r}{1 + s_r^2}, \qquad \sin u = \frac{s_a + s_b s_r}{1 + s_r^2},$$
$$v = a \cos u + b \sin u + r, \qquad w = -a \sin u + b \cos u + s(a, b, r).$$

THEOREM 30. *For a three parameter family of turbines to be a set of tangent turbines of a field, it is necessary and sufficient that the family be not a co-serial set of turbines, that the family be a general set of turbines and finally that the equation*

$$s_a^2 + s_b^2 = 1 + s_r^2$$

*be identically satisfied.*

The field, to which the turbines are the tangent turbines, is unique and it is given by the equations

$$\cos u = \frac{-s_b + s_a s_r}{1 + s_r^2}, \qquad \sin u = \frac{s_a + s_b s_r}{1 + s_r^2},$$
$$v = a \cos u + b \sin u + r, \qquad w = -a \sin u + b \cos u + s(a, b, r).$$

THEOREM 31. *The necessary and sufficient conditions, that a three parameter family of turbines be a set of tangent turbines of a field, are* (1) *that the set of turbines be not a co-serial set of turbines,* (2) *that the set be a general set of turbines, and* (3) *that, with each turbine $T$ of the family and its conjugate turbine $NT$, there be associated two unique elements $E$ and $\bar{E}$ respectively, where the element $E$ is on $T$ and the element $\bar{E}$ is on $NT$, such that every one parameter set of enveloping turbines $\Omega$ of the family has the property that, either the series, to which the turbines are the enveloping turbines, consists of the elements $E$, each of which is on a turbine $T$ of $\Omega$ or that the series,* (*to which the one parameter family of enveloping turbines $N\Omega$, each of which is the conjugate turbine of a turbine of $\Omega$, is the enveloping set of turbines*) *consists of the elements $\bar{E}$, each of which is on a turbine of $N\Omega$.*

THEOREM 32. *If the series, to which the single infinitude of enveloping turbines $N\Omega$* (*or $\Omega$*) *are the enveloping turbines, consists of the element $\bar{E}$* (*or $E$*), *then every element of the series, to which the single infinitude of enveloping turbines $\Omega$* (*or $N\Omega$*) *are the enveloping turbines, is the element of a turbine $T$* (*or $NT$*) *of the enveloping turbines $\Omega$* (*or $N\Omega$*), *which is on the line $q$, such that the tangent line of the curve of centers of the turbines at the center of $T$* (*or $NT$*) *is the bisector of the angle, whose sides are the oriented line of $\bar{E}$* (*or $E$*) *on $NT$* (*or $T$*) *and the line $q$.*

**A characteristic property of whirl transformations.** We begin by considering certain simple operations or transformations on the oriented lineal elements of the plane. A turn $T_a$ converts each element into one having the same point and a direction making a fixed angle $a$ with the original direction. By a slide $S_k$ the line of the element remains the same and the point moves along the line a fixed distance $k$. *These transformations together generate a continuous group of three parameters, which we call the group of whirl transformations and which we denote by $G_3$.* It is easily seen that any whirl transformation may be put in the form [3]

$$T_a S_k T_\beta .$$

The slide $S_k$ is

$$\bar{u} = u, \qquad \bar{v} = v, \qquad \bar{w} = w + k.$$

The turn $T_a$ is

$$\bar{u} = u + \alpha, \qquad \bar{v} = v \cos\alpha + w \sin\alpha, \qquad \bar{w} = -v \sin\alpha + w \cos\alpha.$$

[3] See Kasner, *American Journal of Mathematics*, 1911. The name *whirl* for *TST* was suggested by D. Sole in my seminar. Recently this theory has been extended to spherical geometry by K. Strubecker, *Jahr. d. Math. Ver.*, vol. 44 (1934), pp. 184-198, who suggests the term *turbine-rotation* for whirl.

It is then seen that any whirl transformation may be given by the equations

$$\bar{u} = u + \alpha + \beta, \qquad \bar{v} = v\cos(\alpha+\beta) + w\sin(\alpha+\beta) + k\sin\beta,$$
$$\bar{w} = -v\sin(\alpha+\beta) + w\cos(\alpha+\beta) + k\cos\beta,$$

It is found that the only contact transformations of the set of whirl transformations are

$$T_{\pi/2}S_kT_{-\pi/2}; \qquad T_{\pi/2}S_kT_{\pi/2}.$$

The first represents a dilatation $D_k$; and the second, which may be written $D_kT_\pi$, represents a dilatation accompanied by reversal of orientation.

Our group of whirl transformations may be written in the simple form

$$S_kD_dT_\alpha;$$

and hence any whirl transformation is given in the form

$$\bar{u} = u + \alpha,$$
$$\bar{v} = v\cos\alpha + w\sin\alpha + d,$$
$$\bar{w} = -v\sin\alpha + w\cos\alpha + k.$$

We give now, without proof, a new characteristic property of whirl transformation in terms of the central lines of a field.

THEOREM 33. *For an element transformation to be such, that the central lines of every field $w = f(u, v)$ are identical with the central lines of the corresponding field $\bar{w} = \bar{f}(\bar{u}, \bar{v})$, it is necessary and sufficient that the element transformation be a whirl transformation.*

We remark that the group of whirls is isomorphic to the group of motions. These two groups are commutative and together generate a *new group of six parameters*, of considerable interest in the geometry of elements.

**Extension of Scheffer's theory of isogonal and equi-tangential trajectories.** First we state the following:

THEOREM 34. *If two fields are related by a whirl transformation, then the two conjugate fields are related by a whirl transformation.*

Let $F$ and $G$ be two fields such that $G$ is obtained from $F$ by applying a whirl transformation $W$ to $F$. Then, by means of the above theorem, we know that there exists a whirl transformation $\bar{W}$ such that the two fields $\bar{F}$ and $\bar{G}$, the conjugate fields of $F$ and $G$ respectively, have the property that

$\bar{G}$ is obtained from $\bar{F}$ by applying $\bar{W}$ to $\bar{F}$. We call $\bar{W}$ the conjugate whirl transformation of $W$.

Let $E_0$ and $S_0$ be a fixed element and a fixed series respectively. There exists a unique one parameter family of transformations $T$, which is a subset of the group of whirl transformations $W$, such that any transformation of $T$ carries $E_0$ into an element of $S_0$. It follows that with any element $E$ of the plane there is associated a unique series $S$, such that any transformation of $T$ carries the element $E$ into an element of $S$. We define $S$ to be the quasi-path series of $E$ for the set of transformations $T$. It is seen that the set of quasi-path series for the set of transformations $T$ is at most a three parameter family of series. We denote the totality of quasi-path series for the set of transformations $T$ by $\Sigma$.

We say that the one parameter family of transformations $\bar{T}$ is the conjugate set of transformations of the set of transformations $T$, if each transformation of $\bar{T}$ is the conjugate whirl transformation of a transformation of $T$ and conversely. We denote any quasi-path series of $\bar{T}$ by $\bar{S}$ and the totality of quasi-path series of $\bar{T}$ by $\bar{\Sigma}$. We shall say that two series are conjugate with respect to $T$ or $\bar{T}$, if one is a series of $\Sigma$ and the other is a series of $\bar{\Sigma}$.

Now let us apply $T$ to a one parameter family of curves $F_1$. We then obtain $\infty^1$ new one parameter families of curves, or collectively, a two parameter family of curves $F_2$. Similarly let us apply $\bar{T}$ to the conjugate family of curves $\bar{F}_1$ of the family of curves $F_1$. From Theorems 33 and 34 we obtain:

THEOREM 35. *Consider any one of the quasi-path series $S$ of the set $\Sigma$ connected with $T$. Each element of $S$ determines a curve of the doubly infinite system of curves $F_2$ generated by applying $T$ to any simply infinite system $F_1$. The locus of the centers of the $\infty^1$ circles osculating these curves at these elements is a straight line. Hence these circles touch a certain series $\bar{S}$ of the set $\bar{\Sigma}$ conjugate to $\Sigma$ with respect to $T$.*

THEOREM 36. *According to the previous theorem, the system $F_2$ obtained by applying $T$ to $F_1$ induces a definite correspondence between the set of series $\Sigma$ and the conjugate set $\bar{\Sigma}$. There exists another system $\bar{F}_2$, obtained by applying $\bar{T}$ to $\bar{F}_1$, for which this correspondence is precisely reversed.*

If we place on $T$ the restriction that it be a group of transformations, then the quasi-path series become path series and moreover the path series are turbines all congruent to each other. The set of transformations $\bar{T}$ is also a group of transformations and its path series are turbines, which are the conjugate turbines of the turbines which are the path series of $T$. Thus we

obtain the following two theorems due to Kasner [4] which are themselves generalizations of Scheffer's [5] fundamental theorems on isogonal and equitangential trajectories:

THEOREM 37. *Consider any one of the path turbines $S$ of the set $\Sigma$ connected with the one parameter group of transformations $T$. Each element of $S$ determines a curve of the doubly-infinite system $F_2$ generated by applying $T$ to any simply-infinite system $F_1$. The locus of the centers of the $\infty^1$ circles osculating these curves at these elements is a straight line. These circles touch a certain turbine $\bar{S}$ of the set $\bar{\Sigma}$ conjugate to $\Sigma$.*

THEOREM 38. *According to the previous theorem, the system $F_2$ obtained by applying $T$ to $F_1$ induces a definite correspondence between the set of turbines $\Sigma$ and the conjugate set $\bar{\Sigma}$. There exists another system $\bar{F}_2$, obtained by applying $\bar{T}$ to $\bar{F}_1$, for which this correspondence is precisely reversed.*

In Theorems 37 and 38, if we let $T$ be first the group of turns and then the group of slides, we obtain Scheffer's theorems on isogonal and equitangential trajectories. (Part of Scheffer's first theorem was discovered by Çesaro). The most general families whose central loci are straight lines have been studied by Kasner [6] (velocity families and the dual type). In this connection we may obtain characterizations of both the conformal and the equilong groups.

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[4] *American Journal of Mathematics*, 1911.

[5] *Mathematische Annalen*, 1905.

[6] *Princeton Colloquium*, 1913, 1934, *American Journal of Mathematics*, 1910, and several abstracts in *Bulletin of the American Mathematical Society*, 1930-1935.

# PARALLELISM AND EQUIDISTANCE OF CONGRUENCES OF CURVES OF ORTHOGONAL ENNUPLES.*

By R. M. Peters.

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It is the purpose of this paper to develop some theorems relating to angular and distantial spreads [1] of congruences of curves of orthogonal ennuples lying in an $n$-dimensional Riemannian space $V_n$. We shall be particularly concerned with congruences belonging to a sheaf; that is, to a totality of $\infty^{n-1}$ congruences of which each two cut under a constant angle. It is assumed that the linear element of the $V_n$ is defined by the positive definite quadratic form $ds^2 = g_{ij}dx^i dx^j$, the $x$'s being coördinates of the $V_n$, and the $g_{ij}$'s real analytic functions of the $x$'s. The curves considered are assumed to be real and analytic.

We begin by considering the angular spreads (or associate curvatures, in Bianchi's terminology) of $n$ mutually orthogonal congruences $C_h$, $(h = 1, 2, 3, \cdots, n)$, with respect to any fixed congruence of curves $C$. Let $\lambda_h|^i$ and $\xi^i$ be respectively the unit vectors tangent to the curves $C_h$ and $C$. Then

$$\xi^i = \sum_h \cos\alpha_h\, \lambda_h|^i, \qquad \sum_h \cos^2\alpha_h = 1,$$

where $\alpha_h$ is the angle between the curves $C_h$ and $C$. We denote by $\mu_l|^j$ the angular spread vector, and by $1/r_l$ its length, the angular spread, of the curves $C_l$ with respect to the curves $C$. Then

$$\mu_l|^j = \lambda_l|^j{}_{,i}\, \xi^i = \sum_h \cos\alpha_h\, \lambda_l|^j{}_{,i}\, \lambda_h|^i = \sum_h \cos\alpha_h\, \mu_{lh}|^j, \tag{1}$$

where $\mu_{lh}|^j$ is the angular spread vector of the curves $C_l$ with respect to the curves $C_h$, and $\mu_{ll}|^j$ the first curvature vector of the curves $C_l$. From (1) we conclude

Theorem 1. *If the curves $C_l$ of one congruence are geodesics and are parallel with respect to the curves of every other congruence of the ennuple, then the curves $C_l$ are parallel with respect to the curves of every other congruence in $V_n$; that is, their tangent vectors form a field of parallel unit vectors.*

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* Received February 20, 1937.

[1] For the definition and significance of these terms see Graustein, "The geometry of Riemannian spaces," *Transactions of the American Mathematical Society*, vol. 36, no. 3, p. 555, and Peters, "Parallelism and equidistance in Riemannian geometry," *American Journal of Mathematics*, vol. 57 (1935), pp. 103-111.

THEOREM 2. *If the curves of each congruence of the ennuple are geodesics and are parallel with respect to the curves of every other congruence of the ennuple, then their tangent vectors form n fields of parallel vectors.*[2]

Introducing the coefficients of rotation $\gamma_{pqr}$ of the orthogonal ennuple, we have

$$(2)\qquad \mu_{hk}|^{i}=\lambda_k|^{j}\,\lambda_h|^{i}{}_{,j}=\sum_r \gamma_{hrk}\,\lambda_r|^{i}=-\sum_r \gamma_{rhk}\,\lambda_r|^{i},$$

and

$$(3)\qquad 1/r^2{}_{hk}=\sum_r \gamma^2{}_{rhk},$$

where $1/r_{hk}$ is the angular spread of the curves $C_h$ with respect to the curves $C_k$. Substituting (2) in (1),

$$(4)\qquad \mu_l|^{j}=-\sum_{r,h}\cos\alpha_h\,\gamma_{rlh}\,\lambda_r|^{j},$$

and

$$(5)\qquad 1/r_l{}^2=\sum_{r,h,k}\cos\alpha_h\cos\alpha_k\,\gamma_{rlh}\,\gamma_{rlk}.$$

Now let us consider a second orthogonal ennuple of mutually orthogonal congruences of curves $\bar{C}_h$, $(h=1,2,\cdots,n)$, with unit tangent vectors $\bar{\lambda}_h|^{i}$, and let both ennuples belong to a sheaf. Expressing the vectors $\bar{\lambda}_h|^{i}$ in terms of the vectors $\lambda_k|^{i}$, we have

$$\bar{\lambda}_h|^{i}=\sum_k \cos\beta_{hk}\,\lambda_k|^{i},$$

where $\beta_{hk}$ is the constant angle between the curves $\bar{C}_h$ and $C_k$. Since the two ennuples are orthogonal, the angles $\beta_{hk}$ satisfy the relation

$$(6)\qquad \sum_h \cos\beta_{hk}\cos\beta_{hl}=\delta_l{}^k,$$

$\delta_l{}^k$ being the Kronecker delta.

Let $\bar{\mu}_h|^{j}$ be the angular spread vector and $1/\bar{r}_h$ its length, the angular spread, of the curves $\bar{C}_h$ with respect to the curves $C$. We inquire what relations exist between the two sets of $n$ vectors $\bar{\mu}_h|^{j}$ and $\mu_h|^{j}$, and between the quantities $1/\bar{r}_h$ and $1/r_h$.

$$(7)\qquad \bar{\mu}_h|^{j}=\bar{\lambda}_h|^{j}{}_{,i}\,\xi^i=\sum_k \cos\beta_{hk}\,\lambda_k|^{j}{}_{,i}\,\xi^i=\sum_k \cos\beta_{hk}\,\mu_k|^{j}.$$

Since $\bar{C}_h$ may be any congruence of the sheaf, we can state

[2] The ennuple is then Cartesian and the $V_n$ is Euclidean. See Graustein, *loc. cit.*, p. 564.

THEOREM 3. *The angular spread vector of any congruence of curves* $\bar{C}$ *belonging to a sheaf with respect to an arbitrary congruence of curves* $C$ *is linearly expressible in terms of the angular spread vectors of any orthogonal ennuple of curves* $C_h$ *of the sheaf with respect to the same curves* $C$, *the coefficients of combination being the cosines of the constant angles between the curves* $C_h$ *and* $\bar{C}$.

THEOREM 4. *If the curves of any orthogonal ennuple of congruences of the sheaf are parallel with respect to an arbitrary congruence of curves* $C$, *then the curves of every congruence of the sheaf are parallel with respect to the curves* $C$.

This result combined with Theorem 2 gives

THEOREM 5. *If the curves of each congruence of any orthogonal ennuple of a sheaf are geodesics, and are parallel with respect to the curves of every other congruence of the ennuple, then every congruence of curves of the sheaf is parallel with respect to every congruence of curves in the* $V_n$.

For the angular spread $1/\bar{r}_h$ we have

$$1/\bar{r}_h^2 = \sum_{k,l} \cos \beta_{hk} \cos \beta_{hl}\, g_{ij}\, \mu_k|^i \mu_l|^j.$$

Summing over $h$ and using (6)

$$\sum_h 1/\bar{r}_h^2 = \sum_h 1/r_h^2. \tag{8}$$

In this result is contained a theorem given by Bortolotti:[3] the sum of the squares of the angular spreads of the curves of an orthogonal ennuple of congruences of a sheaf with respect to an arbitrary congruence of curves is the same for every orthogonal ennuple of the sheaf.

We shall show that corresponding facts hold for distantial spread vectors, providing the curves $C$, previously chosen arbitrarily, are required to belong to the sheaf.

Let $\nu_l|^j$ be the angular spread vector of the curves $C$ with respect to the curves $C_l$. Then

$$\begin{aligned}\nu_l|^j &= \lambda_l|^i\, \xi^j{}_{,i} \\ &= \lambda_l|^i \sum_h (\cos \alpha_h\, \lambda_h|^j{}_{,i} + \lambda_h|^j\, \partial \cos \alpha_h/\partial x^i) \\ &= \sum_h (\cos \alpha_h\, \mu_{hl}|^j + \lambda_h|^j\, \partial \cos \alpha_h/\partial s^l),\end{aligned}$$

[3] Bortolotti, "Stelle di congruenze e parallelismo assoluto," *Rendiconti dei Lincei* (6), vol. 9 (1929), pp. 530-538.

where $s^l$ is the arc of the curves $C_l$ and $\partial/\partial s^l$ denotes directional differentiation in the positive direction of the curves $C_l$.

If we denote by $b_l|^j$ the distantial spread vector [4] of the curves $C_l$ and $C$,

$$
\begin{aligned}
b_l|^j &= \mu_l|^j - \nu_l|^j \\
&= \sum_h \left[\cos\alpha_h(\mu_{lh}|^j - \mu_{hl}|^j) - \lambda_h|^j\, \partial \cos\alpha_h/\partial s^l\right] \\
&= \sum_h \left(\cos\alpha_h\, b_{lh}|^j - \lambda_h|^j\, \partial\cos\alpha_h/\partial s^l\right), \qquad (9)
\end{aligned}
$$

where $b_{lh}|^j$ is the distantial spread vector of the congruences of curves $C_l$ and $C_h$. Formula (9) holds when the curves $C$ are the curves of any congruence in $V_n$.

We recall that the distantial spread vector of two congruences vanishes identically if and only if the two congruences lie in a family of two-dimensional surfaces, and the curves of each congruence are equidistant with respect to the curves of the other.[5] Hence we conclude from (9)

THEOREM 6. *If the distantial spread vectors formed for the curves $C_l$ and every other congruence of curves of the ennuple are null vectors, then the curves $C_l$ are equidistant with respect to the congruences of curves $C$ which intersect the curves of all congruences of the ennuple at angles which are constant along the curves $C_l$. In particular, the curves $C_l$ are equidistant with respect to the curves of all congruences belonging to the sheaf.*

If we require that the curves $C$ belong to the sheaf, (9) becomes analogous in form to (1).

$$
b_l|^j = \sum_h \cos\alpha_h\, b_{lh}|^j. \qquad (10)
$$

An ennuple is called a Tchebycheff ennuple [6] if the distantial spread vector formed for each two congruences of the ennuple is a null vector. Hence we have from (10)

THEOREM 7. *If the orthogonal ennuple is an ennuple of Tchebycheff, then the curves of the ennuple are equidistant with respect to every other congruence of curves of the sheaf, and vice versa.*

Returning to the second orthogonal ennuple of curves $\bar{C}_h$ of the sheaf, let $\bar{\nu}_l|^j$ denote the angular spread vector of the curves $C$ with respect to the curves $\bar{C}_l$, where the curves $C$ are again arbitrary.

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[4] For the definition of the distantial spread vector, see Graustein, *loc. cit.*, p. 555.
taken in the order named, then

[5] Graustein, *loc. cit.*, p. 559.

[6] Graustein, *loc. cit.*, p. 563.

$$(11) \qquad \bar{\nu}_l|^j = \bar{\lambda}_l|^i \xi^j{}_{,i} = \sum_k \cos\beta_{lk} \lambda_k|^i \xi^j{}_{,i} = \sum_k \cos\beta_{lk} \nu_k|^j.$$

We note that (11) holds regardless of whether or not the angles $\beta_{lk}$ are constant. That is, (11) gives the relation between the angular spread vectors of an arbitrary congruence of curves $C$ with respect to the curves of two arbitrary orthogonal ennuples.

For the angular spread, $1/\bar{\rho}_l$, of the curves $C$ with respect to the curves $\bar{C}_l$, we have

$$(12) \qquad \begin{aligned} 1/\bar{\rho}_l^2 &= \sum_{h,k} \cos\beta_{lh} \cos\beta_{lk}\, g_{ij}\, \nu_h|^i \nu_k|^j \\ &= \sum_{h,k} \cos\beta_{lh} \cos\beta_{lk} \cos\theta_{hk}/\rho_h\rho_k, \end{aligned}$$

where $\theta_{hk}$ is the angle between the vectors $\nu_h|^i$ and $\nu_k|^j$, and $1/\bar{\rho}_h$ is the angular spread of the curves $C$ with respect to the curves $C_h$. In particular, if we take the curves $\bar{C}_l$ as coincident with the curves $C$, we obtain

$$(13) \qquad 1/\rho^2 = \sum_{h,k} \cos\alpha_h \cos\alpha_k \cos\theta_{hk}/\rho_h\rho_k,$$

$1/\rho$ being the first curvature of the curves $C$.[7]

From (11) and (13) it follows that if the curves $C$ of an arbitrary congruence are parallel with respect to the curves of any orthogonal ennuple of congruences, then the curves $C$ are parallel with respect to the curves of every congruence in the $V_n$; that is, the tangents to the curves $C$ form a field of unit parallel vectors. The curves $C$ are then geodesics.

Summing over $l$ in (12) we obtain

$$(14) \qquad \sum_l 1/\bar{\rho}_l^2 = \sum_l 1/\rho_l^2.$$

This formula gives a theorem corresponding to the one quoted from Bortolotti, the rôles of the curves $C$ and $C_h$ being interchanged.

THEOREM 8. *The sum of the squares of the angular spreads of the curves of an arbitrary congruence with respect to the curves of an orthogonal ennuple is independent of the ennuple chosen.*

Returning to the consideration of distantial spreads we have from (7) and (11) for the distantial spread vector $\bar{b}_l|^j$ of the congruences $\bar{C}_l$ and $C$, in the order named,

[7] This is a generalization of a form of Liouville's formula for geodesic curvature in a $V_2$ given by Graustein, *Transactions of the American Mathematical Society*, vol. 34, no. 3, p. 571.

$$\bar{b}_l|^j = \bar{\mu}_l|^j - \bar{\nu}_l|^j = \sum_k \cos \beta_{lk}\, b_k|^j, \tag{15}$$

where the curves $C_l$ and $\bar{C}_k$ now belong to orthogonal ennuples of the same sheaf so that the angles $\beta_{lk}$ are constants. This result is entirely analogous to formula (7) for the angular spread of the curves $\bar{C}_l$ with respect to the curves $C$, in both cases the curves $C$ being entirely arbitrary. We draw conclusions analogous to Theorems 3, 4, and 5.

THEOREM 9. *The distantial spread vector of any congruence of curves $\bar{C}$ of a sheaf and an arbitrary congruence of curves $C$ is linearly expressible in terms of the distantial spread vectors of any orthogonal ennuple of curves $C_h$ of the sheaf and the same curves $C$, the coefficients of combination being the cosines of the constant angles between the curves $C_h$ and $\bar{C}$.*

THEOREM 10. *If, for an arbitrary congruence of curves $C$ and each congruence of any orthogonal ennuple of the sheaf, the distantial spread vector is a null vector, then the distantial spread vector of $C$ and each congruence of the sheaf is also a null vector.*

Furthermore, using Theorem 7 and requiring that the congruence of curves $C$ belong to the sheaf, we have

THEOREM 11. *If an orthogonal ennuple of the sheaf is an ennuple of Tchebycheff, then every congruence of curves of the sheaf is equidistant with respect to every other congruence of the sheaf.*

We note that this result includes that of Theorem 7.

Let $1/b_l$ and $1/\bar{b}_l$ denote respectively the lengths of the distantial spread vectors $b_l|^j$ and $\bar{b}_l|^j$. Multiplying (15) by $g_{ij}\,\bar{b}_l|^i$, summing over $l$, and using (6) we obtain

$$\sum_l 1/\bar{b}_l{}^2 = \sum_l 1/b_l{}^2, \tag{16}$$

a result analogous to (8) and (14) for angular spreads, which gives the theorem corresponding to Bortolotti's theorem and Theorem 8.

THEOREM 12. *The sum of the squares of the lengths of the distantial spread vectors formed for the curves of an arbitrary congruence of a sheaf and the curves of every congruence of an orthogonal ennuple of the sheaf is independent of the ennuple chosen.*

Instead of the single congruence of curves $C$, let us now consider $n$ mutually orthogonal congruences $C^*_h$, not necessarily belonging to the sheaf,

with unit tangent vectors $\xi_h|^i$. In the previous work we attach an $h$ to each symbol formed with respect to the curves $C$; for example, $\alpha_{kh}$ now denotes the angle between the curves $C_k$ and $C^*_h$. Formula (5) becomes

$$(17)\qquad 1/r^{*2}_{hk} = \sum_{r,p,q} \cos\alpha_{pk} \cos\alpha_{qk}\, \gamma_{rhp}\, \gamma_{rhq},$$

where we have replaced $1/r_h$, the angular spread of the curves $C_h$ with respect to the curves $C$, by $1/r^*_{hk}$, the angular spread of the curves $C_h$ with respect to the curves $C^*_k$. Summing over $k$, we have

$$(18)\qquad \sum_k 1/r^{*2}_{hk} = \sum_{r,p} \gamma^2{}_{rhp},$$

since the angles $\alpha_{pk}$ satisfy the relation

$$\sum_k \cos\alpha_{pk} \cos\alpha_{qk} = \delta_q{}^p.$$

Using (3), (18) becomes

$$(19)\qquad \sum_k 1/r^{*2}_{hk} = \sum_k 1/r^2{}_{hk},$$

and, summing over $h$,

$$(20)\qquad \sum_{h,k} 1/r^{*2}_{hk} = \sum_{h,k} 1/r^2{}_{hk}.$$

Incidentally we note that (19) is essentially identical with (14).

Since we have seen from formula (8) that $\sum_h 1/r^{*2}_{hk}$ is independent of the ennuple of curves $C_h$, and since (20) shows that $\sum_{h,k} 1/r^{*2}_{hk}$ is independent of the ennuple of curves $C^*_k$, we conclude

THEOREM 13. *The sum of the squares of the angular spreads of each congruence of curves of an orthogonal ennuple of a sheaf with respect to the curves of each congruence of an arbitrary orthogonal ennuple is the same for any choice of both ennuples.*

Formula (18) furnishes incidentally a proof of the known fact that the quantity $\sum_{h,k} 1/r^2{}_{hk}$ is the same for every orthogonal ennuple of the sheaf.[8]

If we denote by $b^*_{hk}|^j$ the distantial spread vector of the curves $C_h$ with respect to the curves $C^*_k$, (10) becomes

$$(21)\qquad b^*_{hk}|^j = \sum_l \cos\alpha_{lk}\, b_{hl}|^j,$$

where, we recall, the curves $C^*_k$ now belong to the sheaf. Multiplying by $g_{ij}\, b^*_{hk}|^i$ and summing over $i$, $j$, $k$, and $h$, we have

[8] Graustein, *Transactions of the American Mathematical Society*, vol. 36, no. 3, p. 579, and Bortolotti, *loc. cit.*

(22)
$$\sum_{h,k} 1/b^{*2}_{hk} = \sum_{h,k} 1/b^2{}_{hk},$$

where $1/b^*{}_{hk}$ is the length of the vector $b^*{}_{hk}|^j$.

By (16) $\sum_h 1/b^{*2}_{hk}$ is independent of the ennuple of curves $C_h$ chosen from the sheaf, and by (22) $\sum_{h,k} 1/b^{*2}_{hk}$ is independent of the choice of the ennuple of curves $C^*{}_k$. Hence we have the following

THEOREM 14. *The sum of the squares of the lengths of all the distantial spread vectors of the curves $C_h$ of an orthogonal ennuple of a sheaf formed with respect to all the curves of a second orthogonal ennuple of curves $C^*{}_k$ of the sheaf is independent of the choice of both ennuples.*

Here also we have an indirect proof of the fact that $\sum_{h,k} 1/b^2{}_{hk}$ is the same for every orthogonal ennuple of the sheaf.[9]

Let us now consider the special case when the sheaf contains an orthogonal ennuple of normal congruences. If we take these as the congruences $C_h$, (18) becomes, since $\gamma_{rhp} = 0$ for $r, h, p$ all distinct,

$$\sum_k 1/r^{*2}_{hk} = \sum_r \gamma^2{}_{rhr} + \sum_r \gamma^2{}_{rhh} = \sum_r \gamma^2{}_{hrr} + 1/r^2{}_{hh},$$

$1/r_{hh}$ being the first curvature of the curves $C_h$. Summing over $h$

(23)
$$\sum_{h,k} 1/r^{*2}_{hk} = 2 \sum_h 1/r^2{}_{hh}.$$

Now (2) becomes

$$\mu_{hk}|^i = \gamma_{hkk}\,\lambda_k|^i$$

and hence

$$b_{hk}|^i = \gamma_{hkk}\,\lambda_k|^i - \gamma_{khh}\,\lambda_h|^i,$$
$$1/b^2{}_{hk} = \gamma^2{}_{hkk} + \gamma^2{}_{khh}.$$

Summing over $h$ and $k$,

$$\sum_{h,k} 1/b^2{}_{hk} = 2 \sum_h 1/r^2{}_{hh}.$$

Hence from (20), (22), and (23)

(24)
$$\sum_{h,k} 1/b^{*2}_{hk} = \sum_{h,k} 1/b^2{}_{hk} = 2 \sum_h 1/r^2{}_{hh} = \sum_{h,k} 1/r^2{}_{hk} = \sum_{h,k} 1/r^{*2}_{hk}.$$

Equations (24) show that we have to deal with the following five properties:

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[9] Graustein, *loc. cit.*, and Bortolotti, *loc. cit.*

(A) $1/r_{hh} = 0$, $(h = 1, 2, \cdots, n)$: Curves $C_h$ geodesics.

(B) $1/r_{hk} = 0$, $(h, k = 1, 2, \cdots, n;\ h \neq k)$: Curves $C_h$ parallel with respect to the curves $C_k$.

(C) $1/b_{hk} = 0$, $(h, k = 1, 2, \cdots, n)$: Curves $C_h$ equidistant with respect to the curves $C_k$.

(D) $1/r^*_{hk} = 0$, $(h, k = 1, 2, \cdots, n)$: Curves $C_h$ parallel with respect to the curves $C^*_k$ of any orthogonal ennuple in $V_n$.

(E) $1/b^*_{hk} = 0$, $(h, k = 1, 2, \cdots, n)$: Curves $C_h$ equidistant with respect to the curves $C^*_k$ of any orthogonal ennuple of the sheaf, and vice versa, and each pair of congruences $C_h$ and $C^*_k$ lying in a family of two-dimensional surfaces.

From (24) we conclude

THEOREM 15. *If an orthogonal ennuple of normal congruences of curves $C_h$ has any one of the above properties, then it also has the remaining four, and all congruences of the sheaf consist of geodesics.*[10]

Let us add a few properties to the above list so as to summarize and extend some of our previous results. We rewrite (D) and (E) in slightly different form:

(D) Curves $C_h$, $(h = 1, 2, \cdots, n)$, parallel with respect to the curves of every congruence in $V_n$.

(E) Curves $C_h$, $(h = 1, 2, \cdots, n)$, equidistant with respect to the curves of every congruence in the sheaf, and vice versa.

(F) The curves of every congruence of the sheaf parallel with respect to the curves of every other congruence in $V_n$.

(G) The curves of every congruence of the sheaf equidistant with respect to the curves of every other congruence of the sheaf, and each pair of congruences lying in a family of two-dimensional surfaces.

(H) Curves $C_h$, $(h = 1, 2, \cdots, n)$, all normal.

By Theorem 2, properties (A) and (B) lead to (D); by Theorem 5, these same properties lead to (F), a result superseding the former since (F) includes (D).

By Theorem 7, (C) leads to (E); and by Theorem 11, (C) leads to (G), (G) including (E).

[10] Part of this theorem, to the effect that if (C) holds, then (A) and (B) hold, and conversely, has been proved by Graustein, *loc. cit.*, p. 564, for a non-orthogonal ennuple in which the curves of each two congruences intersect at an angle constant along the curves of both congruences.

We recall that if an orthogonal ennuple is an ennuple of Tchebycheff, it consists of normal congruences.[11] Furthermore, if (B) is valid, then so is (C). Hence we have finally

THEOREM 16. *If either property* (B) *or* (C) *holds, then all the remaining properties hold.*

We shall now denote by $p$ the sheaf containing the curves $C_h$, and consider a second sheaf $p'$ and the relations between the angular and distantial spread vectors of the curves of the two sheaves with respect to an arbitrary congruence of curves $C$ with unit tangent vectors $\xi^i$. Let $C'_h$, $(h = 1, 2, \cdots, n)$, be the curves of any orthogonal ennuple of $p'$, and $\lambda'_h|^i$ their unit tangent vectors. Let $\beta_{hk}$ be the angle between the curves $C'_h$ and $C_k$, where $\beta_{hk}$ is now a variable. Then

$$\lambda'_h|^i = \sum_k \cos \beta_{hk}\, \lambda_k|^i.$$

Denote by $\mu'_h|^i$ the angular spread vector of the curves $C'_h$ with respect to the curves $C$, and by $\nu'_h|^i$ the angular spread vector of the curves $C$ with respect to the curves $C'_h$. Then

$$\begin{aligned} \mu'_h|^i &= \xi^j\, \lambda'_h|^i{}_{,j} \\ &= \xi^j \sum_k (\cos \beta_{hk}\, \lambda_k|^i{}_{,j} + \lambda_k|^i\, \partial \cos \beta_{hk}/\partial x^j) \\ &= \sum_k (\cos \beta_{hk}\, \mu_k|^i + \lambda_k|^i\, \partial \cos \beta_{hk}/\partial s), \end{aligned} \tag{25}$$

where $s$ is the arc of the curves $C$. And

$$\nu'_h|^i = \lambda'_h|^j\, \xi^i{}_{,j} = \sum_k \cos \beta_{hk}\, \nu_k|^i.$$

For $b'_h|^i$, the distantial spread vector of the curves $C'_h$ and $C$, we have

$$b'_h|^i = \mu'_h|^i - \nu'_h|^i = \sum_k (\cos \beta_{hk}\, b_k|^i + \lambda_k|^i\, \partial \cos \beta_{hk}/\partial s). \tag{26}$$

The relations between the lengths of the vectors in question are given by

$$\begin{aligned} \sum_h 1/r'^2_h = \sum_h 1/r_h^2 &+ 2 \sum_{h,k,l} g_{ij}\, \mu_k|^i\, \lambda_l|^j \cos \beta_{hk}\, \partial \cos \beta_{hl}/\partial s \\ &+ \sum_{h,k} (\partial \cos \beta_{hk}/\partial s)^2, \end{aligned} \tag{27}$$

and

$$\begin{aligned} \sum_h 1/b'^2_h = \sum_h 1/b_h^2 &+ 2 \sum_{h,k,l} g_{ij}\, b_k|^i\, \lambda_l|^j \cos \beta_{hk}\, \partial \cos \beta_{hl}/\partial s \\ &+ \sum_{h,k} (\partial \cos \beta_{hk}/\partial s)^2. \end{aligned} \tag{28}$$

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[11] Graustein, *loc. cit.*, p. 563.

The equations (25) through (28) can, of course, be regarded as the relations between the angular and distantial spread vectors of the $n$ congruences of any two orthogonal ennuples with respect to an arbitrary congruence of curves $C$, without bringing in any notion of sheaves.

From (27) and (28) we conclude

THEOREM 17. *The sums of the squares of the lengths of the angular or distantial spread vectors of any orthogonal ennuples of the two sheaves are the same with respect to the curves along which the angles $\beta_{hk}$ are constant.*[12]

In particular, if all the curves $C_h$ of the ennuple of $p$ are parallel with respect to a congruence of curves $C$ along which the angles $\beta_{hk}$ are constant, then all curves of both sheaves are parallel with respect to the curves $C$. A corresponding result can be stated for equidistance.

If the $n$ congruences $C_h$ are normal and consist of geodesics, then (25) becomes

$$\mu'_h|^i = \sum_k \lambda_k|^i \, \partial \cos \beta_{hk}/\partial s,$$

and

$$1/r'^2_{hk} = \sum_k (\partial \cos \beta_{hk}/\partial s)^2. \tag{29}$$

Hence,

THEOREM 18. *The square of the length of the angular spread vector of any congruence of curves $C'$ in a Euclidean $V_n$ with respect to the curves of any other congruence of curves $C$ is equal to the sum of the squares of the directional derivatives in the direction of the curves $C$ of the angles $\beta_k$ between the curves $C'$ and the curves $C_k$, $(k = 1, 2, \cdots, n)$, of an orthogonal ennuple of normal congruences of geodesics.*[13]

If the curves $C$ belong to the sheaf $p$, and again the $n$ congruences of curves $C_h$ are normal and consist of geodesics, then the curves $C$ are also geodesics, and (26) becomes

$$b'_h|^i = \mu'_h|^i = \sum_k \lambda_k|^i \, \partial \cos \beta_{hk}/\partial s,$$

and

$$1/b'^2_h = 1/r'^2_h = \sum_k (\partial \cos \beta_{hk}/\partial s)^2.$$

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[12] A generalization of a result for angular spreads in a $V_2$ given by Graustein, "Parallelism and equidistance in classical differential geometry," *Transactions of the American Mathematical Society*, vol. 34 (1932), p. 570.

[13] This theorem is a generalization of Theorem 14 of Graustein, *loc. cit.*, p. 570.

## ON THE NON-ALTERNATING IMAGES OF LINEAR GRAPHS.*

By Dick Wick Hall.

Let $A$ and $B$ be compact metric spaces and $T(A) = B$ a single valued continuous transformation. Then $T$ is said to be *non-alternating* [1] provided that for no two distinct points $x$ and $y$ of $B$ does the set $T^{-1}(x)$ separate the set $T^{-1}(y)$ in $A$, i. e., there exists no separation $A - T^{-1}(x) = A_1 + A_2$ where $A_1 \cdot T^{-1}(y) \neq 0 \neq A_2 \cdot T^{-1}(y)$. If for each $x \in B$ the set $T^{-1}(x)$ is connected, then $T$ is said to be *monotone*.[2] A connected set $M$ is said to be *cyclic* if it contains no cut point, i. e., if $M - x$ is connected for every $x$ in $M$. If $M$ be a locally connected continuum, and we shall always assume that it is, then a subset $E$ of $M$ will be called a *maximal cyclic subset* if and only if it is not a proper subset of any other cyclic subset of $M$. A subset $E$ of $M$ will be called a *cyclic element* [3] of $M$ provided $E$ is either (a) a maximal cyclic subset of $M$, (b) a cut point of $M$, (c) an end point [4] of $M$. A cyclic element containing more than one point is called a *true cyclic element*. An arc $A$ is said to *span* a point-set $M$ if $A$ has its end points but no other points in common with $M$. Throughout this paper we shall assume that all the linear graphs mentioned are connected, and that all the point-sets considered are imbedded in a three dimensional Euclidean space, since we deal only with 1-dimensional sets and any such set is topologically contained in an $E_3$.

In this paper a study is made of the possible images of a linear graph under a non-alternating transformation. It is shown that: I: A necessary and sufficient condition that a cyclic curve $C$ be the non-alternating image of

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[1] See G. T. Whyburn, *American Journal of Mathematics*, vol. 46 (1934), pp. 294-302. This paper will be referred to as *W*.

[2] This terminology has been suggested by C. B. Morrey. See his paper "The topology of path surfaces," *American Journal of Mathematics*, vol. 57 (1935), pp. 17-50.

[3] See Kuratowski and Whyburn, *Fundamenta Mathematicae*, vol. 16 (1930), pp. 305-350.

[4] A point $p$ is an end point of a locally connected continuum $M$ provided that $M$ contains no simple arc having $p$ as an interior point. See R. L. Wilder, *Fundamenta Mathematicae*, vol. 7 (1925), p. 358. For this particular definition of end point see G. T. Whyburn, *Transactions of the American Mathematical Society*, vol. 29 (1927), Theorem 12, p. 385.

a linear graph is that $C$ be the sum of a finite number of simple arcs; II: If $B$ be the non-alternating image of a linear graph $A$, then every true cyclic element of $B$ is the sum of a finite number of simple arcs; III: Every curve $C$ which is the sum of a finite number of simple arcs is the non-alternating image of a linear graph.

THEOREM 1. *If $A$ be a linear graph and $T(A) = B$ is non-alternating, then every true cyclic element of $B$ is the sum of a finite number of simple arcs.*

*Proof.* By $(W, 3.4)$, if $E_b$ be any true cyclic element of $B$ there exists a non-alternating transformation $W(A) = E_b$ such that none of the sets $W^{-1}(x)$ separate $A$; and by $(W, 3.5)$, there exists a true cyclic element $E_a$ of $A$ such that $W(E_a) = E_b$. Then none of the sets $W^{-1}(x)$ separate $E_a$, since any set which did so would also separate $A$ which is impossible.

Now since $E_a$ is a true cyclic element of a linear graph we may write $E_a = \sum_1^k \bar{A}_i$, where $k$ is finite and $\bar{A}_i = a_i b_i$ is the closure of a free arc $A_i$. Then $W$ is monotone on $A_i$, for all $i$. Otherwise, for some $x$ in $E_b$, $W^{-1}(x)$ would separate $E_a$, which is impossible. Two cases may arise: (a) If $W(a_i) = W(b_i) = c_i$ for any $i$, then $\bar{A}_i$ maps into a single closed curve having only the point $c_i$ in common with the rest of $E_b$. Consequently, since $E_b$ is cyclic we must have $W(\bar{A}_i) = E_b$, a simple closed curve. Hence $E_b$ is the sum of two simple arcs and the theorem follows. (b) If $W(a_i) \neq W(b_i)$ for any $i$, then $W$ is monotone on $\bar{A}_i$ for all $i$. Thus $W(\bar{A}_i) = B_i$ is a simple arc, and $E_b = \sum_1^k B_i$, which is the theorem.

LEMMA 1. *If $K$ be a cyclic curve such that there exists a linear graph $H$ and a non-alternating transformation $T(H) = K$, and if $A$ be any simple arc such that $K + A$ is cyclic, then there exists a simple arc $B$ spanning $H$, and a non-alternating transformation $Z(H + B) = K + A$.*

*Proof.* We may assume that $H$ is cyclic, by $(W, 3.5)$. Let $a'$, $a''$ be the end points of $A$, and $b'$, $b''$ any points of $T^{-1}(a')$ and $T^{-1}(a'')$ respectively. Let $B$ be a simple arc spanning $H$ and having $b'$ and $b''$ as end points. Define $Z$ so that it is identical with $T$ on $H$, while on $B$ it is a homeomorphism sending $B$ into $A$ and such that $Z(b') = a'$, and $Z(b'') = a''$. Then $Z(H + B) = K + A$.

Moreover, $T$ is non-alternating. Otherwise, we could find two points $x, y$ in $K + A$ such that $Z^{-1}(x)$ separated $Z^{-1}(y)$ in $H + B$. Now (1): $x$ cannot lie in $A - K \cdot A$. Otherwise, $Z^{-1}(x)$, which is a single point since $Z$ is one-to-one on $B - B \cdot H$, would separate the cyclic set $H + B$, which is impossible. (2) $y$ cannot lie in $A - K \cdot A$, since $Z$ is one-to-one on $H - B \cdot H$, and no single point can be separated. (3) Consequently, both $x$ and $y$ must lie in $K$. Since $Z^{-1}(x)$ separates $Z^{-1}(y)$ in $H + B$, it follows that $Z^{-1}(y)$ contains two points $y'$ and $y''$ such that $Z^{-1}(x)$ separates $y'$ and $y''$ in $H + B$. Now not both $y'$ and $y''$ may lie in $H$, since $T$ is non-alternating on this set. Hence we may assume that $y'$, say, lies in $B - B \cdot H$. By the definition of $Z$, $B \cdot Z^{-1}(x)$ is a single point, hence $Z^{-1}(x)$ cannot separate $y'$ from both $b'$ and $b''$ say not from $b'$. Then, since $Z^{-1}(x)$ separates $y'$ from $y''$, it must separate $b'$ from $y''$. But we may assume that $y'' \epsilon H$, and hence $Z^{-1}(x)$ must separate two points of $H$. Thus $T^{-1}(x)$ must separate two points of $H$; and consequently, by $(W, 1.41)$, $x$ is a cut point of $K + A$, which is a contradiction.

Therefore, $T$ is non-alternating, and the lemma is proved.

THEOREM 2. *A necessary and sufficient condition that a cyclic curve $C$ be the non-alternating image of a linear graph is that $C$ be the sum of a finite number of simple arcs.*

*Proof.* Necessity: This is immediate from Theorem I. Sufficiency: Let $C = \sum_1^n A_i$, where $n$ is finite and each $A_i$ is a simple arc. Let the $2n$ end points of these simple arcs be denoted by $a_1, a_2, \cdots, a_{2n}$, where an end point is counted once for each arc of which it is an end point. Since $C$ is cyclic, it contains a simple closed curve $K_1$ passing through $a_1$ and $a_2$. If $K_1$ does not contain all the points $a_i$, let $a_j$ be any one of these points which it does not contain. Then, by the three point theorem [5] we may find a simple arc spanning $K_1$ and containing $a_j$. Thus we have found a cyclic linear graph containing $a_1, a_2, a_j$. Repeating this process a finite number of times we shall obtain a cyclic linear graph $K$ which is a subset of $C$ and which contains all of the points $a_i$. Thus $K + A_i$ is cyclic for every $i$. We now write $C = K + \sum_1^n A_i$, and the theorem follows at once by $n$ applications of Lemma 1, and the addition of one of the arcs $A_i$ to $K$ at each step.

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[5] See W. L. Ayres, *Bulletin Académie Polonaise Science et Lettres* (1928), pp. 127-142.

By a *$\theta_n$-curve* we shall mean a curve expressible as the sum of $(n+2)$ simple arcs having the same end points but otherwise disjoint by pairs.

Using the same construction as that employed in Theorem 2, the following lemma is immediate.

LEMMA 2. *Every cyclic curve $C$ expressible as the sum of $n$ simple arcs having the same end points is the non-alternating image of a $\theta_n$-curve.*

It can be shown that no $\theta_2$ curve is the image of a $\theta_1$-curve under a non-alternating transformation. Consequently, since a $\theta_2$-curve $A$ is easily expressible as the sum of two simple arcs having as end points two points interior to different free arcs of $A$, it follows that the $n$ in Lemma 2 cannot be reduced. For by $(W, 4.6)$ $A$ is not the non-alternating image of a $\theta_0$-curve, that is, of a simple closed curve, and hence not the non-alternating image of any $\theta_k$-curve for $k < 2$.

Our next lemma will remove the restriction that $C$ be cyclic.

LEMMA 3. *Every curve $C$ which is the sum of $n$ simple arcs $A_i$ $(i = 1, 2, \cdots, n)$ having the same end points $a$ and $b$ is the non-alternating image of a $\theta_n$ curve.*

*Proof.* For notational reasons we give the proof for the case $n = 2$, since the general case follows in precisely the same way.

Since every arc in $C$ joining $a$ and $b$ must be in every $A$-set [6] containing $a$ and $b$ it follows at once that $C = C(a, b)$, that is, $C$ is a simple cyclic chain joining $a$ and $b$.

Let $K$ be the set of all points separating $a$ and $b$ in $C$. Then, since $C$ is a simple cyclic chain, we may write $C = (K + a + b) + \Sigma C_i$, where each $C_i$ is a true cyclic element of $C$.

Define a $\theta_2$-curve $H = \sum_1^4 a'x_ib'$, where the $a'x_ib'$ are simple arcs having the same end points but otherwise disjoint by pairs. We shall prove that $C$ is the image of $H$ under a non-alternating transformation.

Let $axb$ be any simple arc joining $a$ and $b$ in $C$. Then, by the definition of $K$, it follows that $K$ is a subset of $axb$. Let $Z_i(a'x_ib') = axb$ be a homeomorphism defined on $a'x_ib'$ (for each $i$) and sending $a'$ and $b'$ into $a$ and $b$, respectively. Let $K_i = Z_i^{-1}(K + a + b)$. Define a transformation $Z$ of $H$

[6] For definitions of the new terms used see G. T. Whyburn, *American Journal of Mathematics*, vol. 50 (1928), pp. 167-194. In connection with $A$-sets see also Kuratowski and Whyburn, *loc. cit.*

into a new curve $H'$ as follows: (i) $Z$ is identical with $Z_i$ on $K_i$, (ii) $Z$ is a homeomorphism on $H - \sum_1^4 K_i$. Then $H'$ will coincide with $C$ at all points of $(K + a + b)$. Moreover, by the definition of $Z$ every true cyclic element of $H'$ will consist of four simple arcs disjoint by pairs except for their end points and joining two points of $(K + a + b)$ in $C$. Thus every true cyclic element of $H'$ will be a $\theta_2$-curve. But every true cyclic element of $C$ is the sum of two simple arcs having their end points in common and hence, by Lemma 2, every such true cyclic element is the non-alternating image of a $\theta_2$-curve. From the proof of Theorem 2 it follows that we may pick non-alternating transformations sending the true cyclic elements of $H'$ into those of $C$ in such a manner that they will map the points of $H'_i\, K$ (where $H'_i$ is the given true cyclic element) into themselves. Thus we may define a transformation $W(H') = C$ to be the identity transformation on $K$, and to send each true cyclic element of $H'$ into the corresponding true cyclic element $C_i$ of $C$ (namely the one containing the same points of $(K + a + b)$) in a non-alternating manner.

Let $H'_i$ be any true cyclic element of $H'$ and suppose that $W(H'_i) = C_i$. Then, by $(W, 1.41)$, since $W$ is non-alternating on $H'_i$ and $C_i$ is cyclic, it follows that for no $x$ in $C_i$ does $W^{-1}(x)$ separate $H'_i$. Also, by definition of $W$, the images of the end points of each free arc of $H'_i$ are distinct. Thus $W$ is monotone on the closure of each free arc of $H'_i$.

Let $axb$ be any simple arc joining $a$ and $b$ in $H'$. Then $W$ is monotone on $axb$. To prove this it is sufficient to show that if $p$ and $q$ be any two points on $axb$ such that $W(p) = W(q)$, and if $z$ be any point between $p$ and $q$ on $axb$, then $W(z) = W(p) = W(q)$. Since $W$ is the identity transformation on $K$, $p$ and $q$ cannot both lie in $K$, so we may assume that $p$ lies on a free arc of some true cyclic element $H'_j$ of $H'$. Then if $q$ lies in $H'_j$ our assertion is established since $W$ is monotone on the closure of each free arc of this set. If $q$ is not in $H'_j$, then, since $W$ maps disjoint cyclic elements of $H'$ into disjoint cyclic elements of $C$, it follows that $q$ lies in some $H'_k$, where $H'_j \cdot H'_k = y$, a single point (which may or may not be $q$). Let $pyq$ be a simple arc joining $p$ to $q$ in $H'_j + H'_k$. Then, since $W$ is monotone on the closure of each free arc of both $H'_j$ and $H'_k$, we have $W(pyq) = W(y)$, so that in particular, $W(z) = W(y) = W(p) = W(q)$, which proves $W$ monotone on $axb$.

By definition the transformation $Z(H) = H'$ is monotone on the closure of each free arc of $H$, hence by $(W, 2.2)$, if we define a transformation $T = WZ$ it follows that $T$ is monotone on the closure of each free arc of $H$.

By definition we have $T(H)=W(H')=C$, so that the lemma will be established if we show that $T$ is non-alternating on $H$.

The set $H$ is locally connected, hence by $(W, 1.5)$, in order to show that $T(H)=C$ is non-alternating, it is sufficient to show that for each point $q \epsilon C$ and each component $K$ of $C-q$, the set $T^{-1}(K)$ is connected. Letting $q$ be any point of $C$, two cases must be considered.

*Case* 1. $q$ is a cut point of $C$.

Then $q$ is distinct from both $a$ and $b$. Recalling that by definition $H=\sum_1^4 a'xb'$, it follows that $B_i \equiv T(a'x_ib')=ax_ib$ is a simple arc since $T$ is monotone on the closure of each of the simple arcs $a'x_ib'$. Since $q$ cuts $C$ it lies on all the arcs $B_i$, so we may write $B_i=ax_iq+qy_ib$. Then since $C=\sum_1^4 B_i$, we have $C-q=\sum_1^4 (ax_iq-q)+\sum_1^4 (qy_ib-q)\equiv M_1+M_2$. Then $M_1$ and $M_2$ are both connected and closed in $C-q$. Consequently, since $C-q$ is disconnected, these sets are mutually separated, and hence components of $C-q$.

Let a transformation $T_i$ be defined as identical with $T$ on $a'x_ib'$, and undefined elsewhere. Then $T_i$ is monotone and $T(a'x_ib'=B_i$. Thus for every $i$ the sets $T_i^{-1}(ax_iq-q)$ and $T_i^{-1}(qy_ib-q)$ are connected. But these are precisely the sets

$$T^{-1}(ax_iq-q)\cdot(a'x_ib'), \tag{1}$$

$$T^{-1}(qy_ib-q)\cdot(a'x_ib'), \tag{2}$$

respectively, so that the sets (1) and (2) are connected for all $i$. Thus

$$a'x_ib'-T^{-1}(q)=T^{-1}(ax_iq-q)\cdot(a'x_ib')+T^{-1}(qy_ib-q)\cdot(a'x_ib'),$$

and therefore,

$$\sum_1^4 a'x_ib'-T^{-1}(q)=\sum_1^4 T^{-1}(ax_iq-q)\cdot(a'x_ib')+\sum_1^4 T^{-1}(qy_ib-q)\cdot(a'x_ib'),$$

so that

$$H-T^{-1}(q)=\sum_1^4 T^{-1}(ax_iq-q)\cdot(a'x_ib')+\sum_1^4 T^{-1}(qy_ib-q)\cdot(a'x_ib')$$
$$\equiv N_1+N_2,$$

and from (1) and (2), it follows that $N_1$ and $N_2$ are connected. They are evidently disjoint and closed in their sum, so they are mutually separated.

For any point $x \epsilon N_1$, we have $T(x)\ \epsilon \sum_1^4 (ax_iq-q)\subset M_1$, so that $x \epsilon T^{-1}(M_1)$, hence $N_1 \subset T^{-1}(M_1)$, and similarly $N_2 \subset T^{-1}(M_2)$.

Moreover, for any point $x \in T^{-1}(M_1)$ we have $T(x) \in M_1$ so that $T(x)$ non $\in M_2$. Consequently, since $N_2 \subset T^{-1}(M_2)$, we have $x$ non $\in N_2$; therefore $x \in N_1$, so that $T^{-1}(M_1) \subset N_1$. Thus $N_1 = T^{-1}(M_1)$, and similarly $N_2 = T^{-1}(M_2)$, so both of these sets are connected and the lemma follows for Case 1.

*Case* 2. $q$ is a non-cut point of $C$.

We have again $B_i = T(a'x_ib')$. If $q \in B_i$ write $B_i = ax_ib = ax_iq + qy_ib$. Otherwise, write $B_i = ax_ib$. Let the $a'x_ib'$ be numbered so that for some integer $j$, $q \in B_i$, $(1 \leq i \leq j)$, $q$ non $\in B_i$, $(i > j)$. (If $q \in B_i$ for every $i$, then we have, of course, $j = 4$ since $H$ contains but four free arcs by hypothesis). Then

$$C - q = \sum_{j+1}^{4} ax_ib + \sum_{1}^{j} (ax_iq - q) + \sum_{1}^{j} (qy_ib - q).$$

Define

$$M = \sum_{1}^{j} T^{-1}(ax_iq - q) \cdot (a'x_ib'), \qquad N = \sum_{1}^{j} T^{-1}(qy_ib - q) \cdot (a'x_ib'),$$
$$Z = \sum_{j+1}^{4} T^{-1}(ax_ib) \cdot (a'x_ib').$$

By the reasoning used in Case 1, each of the sets $M$, $N$, and $Z$ is vacuous or connected.

Assume, first, that $Z$ is non-vacuous. Then $q$ is distinct from both $a$ and $b$ so that $M$ contains $a'$, $N$ contains $b'$, and $Z$ contains both of these points. Therefore, since each of these sets is connected we have that $(M + N + Z)$ is a connected set.

Secondly, if $Z$ is vacuous, then $q$ occurs on every simple arc $B_i$. If $q = a$, then $M = 0$, and both $N$ and $Z$ contain $b'$, so that $(M + N + Z)$ is connected. Similarly, if $q = b$, then $N = 0$, both $M$ and $Z$ contain $a'$, and, consequently, $(M + N + Z)$ is connected. If $q$ is distinct from both $a$ and $b$, then, since $q$ does not cut $C$, there exist two simple arcs $B_j$ and $B_k$ in $C$ and a point $p$ in $C$ which precedes $q$ on the arc $B_k$ but follows it on the arc $B_j$. Then $M$ and $N$ have $T^{-1}(p)$ in common so that $(M + N)$ and hence $(M + N + Z)$ is connected. Therefore, in every event, $(M + N + Z)$ is a connected set.

But, since $C = \Sigma\, T(a'x_ib')$, we have that $T^{-1}(C - q) = M + N + Z$, so that this set is connected. Therefore, for any point $q$ of $C$ and every component $K$ of $C - q$, the set $T^{-1}(K)$ is connected. Consequently, by ($W$, 1.5), the transformation $T(H) = C$ is non-alternating, as was to be proved.

LEMMA 4. *Suppose that $A$, $C$, and $A + C$ are connected and that $A \cdot C = p$, a single point. Let $T'(A) = B$ and $T^2(C) = D$ be non-alternating*

*transformations such that $T' = T^2$ on $AC$. Define a transformation $T = T'$ on $A$, $T = T^2$ on $C$. Then a necessary and sufficient condition that $T$ be non-alternating on $(A + C)$ is that $B \cdot D = T(p)$.*

*Proof.* The condition is clearly necessary. Otherwise, the inverse of some point $x$ of $(B + D) - p$ would intersect both $A - p$ and $C - p$, so that this inverse would be separated by $T^{-1}T(p)$ in $(A + C)$ which would make $T$ alternate on this set.

The condition is also sufficient. Otherwise, there exist two points $y'$, $y''$ in $(A + C)$, with $T(y') = T(y'')$, and a point $x$ in $B + D$ such that $T^{-1}(x)$ separates $y'$ from $y''$ in $(A + C)$. Then (i) if $y'$ and $y''$ both lie in $A$ or $C$, we have a contradiction to the fact that $T$ is non-alternating on these respective sets; (ii) if $y' \epsilon A$, $y'' \epsilon C$, or conversely, then, since $T(y') = T(y'')$ and $B \cdot D = p$, we have $T(y') = T(y'') = T(p)$, and thus $p$ non $\epsilon\, T^{-1}(x)$. Hence, by the definition of $T$, $T^{-1}(x)$ cannot separate either $y'$ or $y''$ from $p$; so that it cannot separate $y'$ from $y''$, and the lemma follows.

LEMMA 5. *If $C = \sum_1^n A_i$ be connected, where $n$ is finite and each $A_i$ is a connected $A$-set (in some locally connected continuum $S$) which is the non-alternating image of a linear graph and where for no $i$, $j$ $(i \neq j)$ does $A_i \cdot A_j$ contain more than one point, then $C$ is the non-alternating image of a linear graph.*

*Proof.* By hypothesis there exists a set of disjoint linear graphs $H_i$ $(i = 1, 2, \cdots, n)$ and a set of non-alternating transformations $T_i(H_i) = A_i$.

Consider the $A$-set $A_1$. Since $C$ is connected, at least one of the other sets $A_i$, say $A_2$, must intersect it. Then $A_1 \cdot A_2 = p_{12}$, a single point by hypothesis. Let $p_1$, $q_1$ be any two points of $T_1^{-1}(p_{12})$ and $T_2^{-1}(p_{12})$, respectively, and translate $H_2$ until it has the single point $p_1 = q_1$ in common with $H_1$. Then, by Lemma 5, the transformation $T_{12}$ which is $T_1$ on $H_1$ and $T_2$ on $H_2$ is non-alternating, and $T_{12}(H_1 + H_2) = A_1 + A_2$.

Repeating this process we may add on one $A$-set at a time, and at each stage secure a non-alternating transformation sending $\sum_1^k H_i$, for example, into $\sum_1^k A_i$ at the $k$-th stage. It follows at once that $A_k$ cannot have two points in common with $\sum_1^{k-1} A_i$, since if $p$ and $q$ were any two such points they would, by hypothesis, lie in different $A$-sets of the above sum. Then, using the fact that each of the $A$-sets is connected and locally connected, we could construct a simple arc joining $p$ and $q$ and not lying wholly in $A_k$, which is contradictory

to the definition of an $A$-set. Thus the extension may be made in exactly the same way at every stage and the lemma follows.

LEMMA 6. *Every simple cyclic chain which is the sum of a finite number of simple arcs is the non-alternating image of a linear graph.*

*Proof.* Let $C(a, b)$ be the simple cyclic chain which is the sum of $j$ simple arcs by hypothesis, and let $a_i$ $(i = 1, 2, \cdots, 2j)$ be the endpoints of these simple arcs. Then we may find $2j$ cyclic elements of $C(a, b)$ whose sum contains all the points $a_i$. Let $n$ be the number of these cyclic elements which are distinct, and number them $K_i$ in the order in which they occur from $a$ to $b$. Since $C(a, b)$ is a simple cyclic chain, let it be expressed in the form $C = (K + a + b) + \Sigma C_i$, where the $C_i$ are its true cyclic elements, and $K$ consists of all those points separating $a$ and $b$ in $C(a, b)$. Let $axb$ be any simple arc joining $a$ and $b$ in $C(a, b)$, and let $x_i$, $y_i$ be respectively its first and last intersections with the set $K_i$. Evidently $x_i = y_i$ if $K_i$ is degenerate. Then $x_1 = a$, $y_n = b$, while all of the points $x_i$, $y_i$ which are distinct from these two are points of $K$. Consequently, the choice of the points $x_i$, $y_i$ is independent of the arc $axb$ we used. Define $M_{2i-1} = C(x_i, y_i)$, $M_{2i} = C(y_i, x_{i+1})$ as simple cyclic chains in $C$. Some of the sets $M_i$ may be degenerate.

Evidently the chains $M_i$ as constructed are finite in number and $C(a, b) = \sum_{1}^{2n-1} M_i$. Moreover, these sets are $A$-sets, by definition, and for any $i, j$ $(i \neq j)$, $M_i \cdot M_j$ contains not more than one point. Our lemma will thus follow by Lemma 6 if we show that, for every $i$, $M_i$ is the non-alternating image of a linear graph. This follows at once for every $M_{2i-1}$ by Theorem II, since every such set is a $K_i$, hence cyclic, and is the sum of a finite number of simple arcs, since $C$ is. Now $M_{2i}$ is a cyclic chain joining $y_i$ to $x_{i+1}$, and both these points belong to $K$. Moreover, with the possible exception of these two points, $M_{2i}$ cannot contain end points of any of the simple arcs which go to make up $C(a, b)$; whence $M_{2i}$ is the sum of a finite number of simple arcs joining $y_i$ and $x_{i+1}$. The lemma is then an immediate consequence of Lemma 3.

THEOREM III. *Every curve $C$ which is the sum of a finite number of simple arcs is the non-alternating image of a linear graph.*

*Proof.* Let $C$ be the sum of $j$ simple arcs. Then these arcs have not more than $2j$ distinct end points; whence $C$ contains not more than $2\,j$ nodes,[7] since every node of $C$ must evidently contain an end point of at least one of the $j$

[7] See G. T. Whyburn, *American Journal of Mathematics*, vol. 50 (1928), p. 178.

simple arcs of which $C$ is the sum. Let there be $h$ nodes in $C$ and call them $E_i$ $(i = 1, 2, \cdots, h)$. Let $p_i$ be any non-cut point contained in $E_i$. Define $M_2 = C(p_1, p_2)$. Let $p_1xp_3$ be any simple arc joining $p_1$ to $p_3$ and let $q_3$ be the last intersection of this arc with $M_2$. Define $M_3 = C(p_3, q_3)$. In general, if $M_{k-1}$ has been defined let $p_1xp_k$ be a simple arc joining $p_1$ to $p_k$ in $C$ and let $q_k$ be its last intersection with $\sum_2^{k-1} M_i$. Define $M_k = C(p_k, q_k)$.

Evidently, $C = \sum_2^h M_i$, and each $M_i$ is the non-alternating image of a linear graph by virtue of Lemma 6. Moreover, the conditions of Lemma 5 are obviously fulfilled so that the theorem follows by virtue of that lemma.

The converse of Theorem III is false. For, let $D$ be any dendrite which is not the sum of a finite number of simple arcs (and such a dendrite may easily be constructed). Then, by ($W$, p. 301), $D$ is a boundary curve, and hence, by ($W$, 4. 6), $D$ is the non-alternating image of a circle, which is certainly a linear graph. Therefore, the non-alternating image of a circle need not be the sum of a finite number of simple arcs.

UNIVERSITY OF VIRGINIA.

# REPRESENTATIONS IN CERTAIN PURE FORMS OF DEGREES HIGHER THAN THE SECOND.*

By E. T. Bell.

**1. Introduction.** The method of Lagrange [1] for finding parametric solutions of certain diophantine equations may be taken as the point of departure for obtaining representations of integers in special forms of degree $\geqq 2$. This will be illustrated by starting from the equation

$$(1.1) \qquad a^3 + b^3 + pc^3 = 0, \qquad pabc \neq 0,$$

in which $p$ is a constant integer. Numerical examples are given in § 5. Another method is indicated in § 6, with examples. *Throughout the paper, $x, y, z, u, v, w$ denote real or complex variables, other small letters, rational integers.* Without loss of generality $a, b$ may be taken coprime, $(a, b) = 1$, and any divisor $r^3$ of $p$ may be absorbed in $c$ if desired. It was shown by Lucas [2] that if $\{d\}$ denotes $d$ divided by the greatest cube divisor of $d$, a necessary and sufficient condition that (1.1) have a solution is that $p$ be of the form $\{st(s+t)\}$.

If $(a, b, c) = (a_n, b_n, c_n)$ is any solution of (1.1), $(a_{n+1}, b_{n+1}, c_{n+1})$ is also a solution, where

$$(1.2) \quad a_{n+1} = a_n(a_n^3 + 2b_n^3), \; b_{n+1} = -b_n(2a_n^3 + b_n^3), \; c_{n+1} = c_n(a_n^3 - b_n^3),$$

and if $(a_{n+1}, b_{n+1}, c_{n+1}) \neq (ka_n, kb_n, kc_n)$, the two solutions are said to be distinct.[3] A parametric solution of (1.1), due to Lucas, is

$$(1.3) \quad \begin{aligned} a &= x^3 - y^3 + 6x^2y + 3xy^2, & b &= -x^3 + y^3 + 3x^2y + 6xy^2, \\ c &= 3(x^2 + xy + y^2), & -p &= xy(x+y); \end{aligned}$$

if $a, b, c, p$ are integers, $x, y$ run through all integers. By means of (1.3),

* Received April 30, 1937.

[1] Supplement to Euler's *Algebra*; see also R. D. Carmichael's *Diophantine Analysis.*

[2] E. Lucas, *American Journal of Mathematics*, vol. 2 (1879), p. 184. The paper by L. Holzer in *Journal für Mathematik*, vol. 159 (1928), pp. 93-100, discusses the same equation, and obtains (incidentally) some of the results of Lucas and Sylvester, whose papers on the subject appear to have been overlooked by the author.

[3] Lucas asserts, *loc. cit.*, p. 185, that if $|a| \neq |b|$, there is an infinity of distinct solutions. This has not been proved.

Lucas solved (in particular) (1.1) with $p=6$, which Legendre had mistakenly asserted to be unsolvable in integers.

**2. Identities from (1.1).** Let $(a, b, c)$ be any solution of (1.1), without the conditions $(a, b)=1$, $|c|$ free of cube divisors $>1$, so that

$$a^3+b^3+pc^3=0, \qquad pabc\neq 0. \tag{2.1}$$

We shall determine the parameters $w, x, y, z$ so that, identically in $w, x, y, z$,

$$(aw+x)^3+(bw+y)^3+p(cw+z)^3\equiv 3Aw+B, \tag{2.2}$$

with $A, B$ independent of $w$.

LEMMA 1. *If $x, y, z$ are such that*

$$a^2x+b^2y+pc^2z=0, \tag{2.3}$$

*where $(a, b, c)$ is a particular solution of* (2.1), *then identically in $w, x, y, z$,*

$$\begin{aligned} pc^4[(aw+x)^3+(bw+y)^3+p(cw+z)^3] \\ \equiv -(bx-ay)^2(3abcw+bcx+cay+abz). \end{aligned} \tag{2.4}$$

For, from (2.1), (2.2) we get (2.3) and $A=ax^2+by^2+pcz^2$, $B=x^3+y^3+pz^3$; whence, eliminating $z$ by (2.3) we have

$$\begin{aligned} -pc^3A &= ab(bx-ay)^2, \\ p^2c^6B &= (bx-ay)^2[b(2a^3+b^3)x+a(a^3+2b^3)y]. \end{aligned}$$

Hence, if (2.2) be multipled throughout by $p^2c^6$, the right becomes $(bx-ay)^2F$,

$$\begin{aligned} F &\equiv -[3abpc^3w-(2a^3+b^3)bx-(a^3+2b^3)ay]. \\ F &= -[3abpc^3w-(a^3-pc^3)bx-(b^3-pc^3)ay], \\ &= -[pc^3(3abw+bx+ay)-ab(a^2x+b^2y)], \\ &= -pc^2(3abcw+bcx+cay+abz); \end{aligned}$$

which completes the proof of (2.4).

LEMMA 2. *If, without loss of generality, $(a, b)=1$ in* (2.1), *so that (integers) $r, s$ may be determined such that*

$$a^2r+b^2s=-1, \tag{2.5}$$

*then, identically in $w, u, v$,*

$$\begin{aligned} (aw+prc^2u+b^2v)^3+(bw+psc^2u-a^2v)^3+p(cw+u)^3 \\ \equiv -pM^2(3abcw+R), \end{aligned} \tag{2.6}$$

*where* $M \equiv (br - as)u - cv,$

$$R \equiv (prbc^3 + psac^3 + ab)u + (b^3 - a^3)v.$$

For, from (2.5), (2.3) we get

$$(x, y, z) = (prc^2u + b^2v, psc^2u - a^2v, u),$$

with $(u, v)$ arbitrary, and the result follows by (2.1), (2.4).

In (2.6) we now choose $(u, v) = (m, n)$, $m$ $n$ integers, and reduce the right numerically. If $[y]$ is the greatest integer in $y$, we write

$$(2.7) \qquad G \equiv \left[\frac{|R|}{3\,|abc|}\right], \qquad \eta \equiv \operatorname{sgn} R, \qquad \epsilon \equiv \operatorname{sgn}(abcR)$$

after having replaced $(u, v)$ by $(m, n)$ in $R$, where $\operatorname{sgn} y$ denotes $1, 0, -1$ according as $y > 0$, $y = 0$, $y < 0$. Then

$$(2.8) \qquad |R| = 3\,|abc|\,G + \rho, \qquad 0 \leqq \rho < 3\,|abc|.$$

Replacing $w$ by $w - \epsilon G$ in $3\,abcw + R$ we find $3abcw + \eta\rho$.

Lemma 3. *With $G$, $\rho$ as in* (2.7), (2.8),

$$(aw + prc^2m + b^2n - \epsilon aG)^3 + (bw + psc^2m - a^2n - \epsilon bG)^3$$
$$+ p(cw + m - \epsilon G)^3 \equiv -pM^2(3abcw + \eta\rho),$$

*identically in $w$, where*

$$M \equiv (br - as)m - cn,$$

*and $a$, $b$, $c$, $r$, $s$ are as in Lemma* 2.

Returning to (2.6) we recall that $(r, s)$ are defined by (2.4). Let

$$(2.9) \qquad (br - as, c) = g.$$

Then integers $h$, $k$ may be found such that

$$(2.10) \qquad (br - as)h - ck = g,$$

and (2.12) is a solution of

$$(2.11) \qquad (br - as)u - cv = gx,$$

$$(2.12) \qquad u = hx + cy, \qquad v = kx + (br - as)y.$$

LEMMA 4. *With $r, s, g, h, k$ as in (2.4), (2.9), (2.10),*

$$-pg^2x^2(3abcw + R) \equiv X^3 + Y^3 + pZ^3$$

*identically in $w, x, y$, where*

$$\begin{aligned}
X &\equiv aw + (prc^2h + b^2k)x + ay, \\
Y &\equiv bw + (psc^2h - a^2k)x + by, \\
Z &\equiv cw + hx + cy, \\
R &\equiv [h(prbc^3 + psac^3 + ab) + k(b^3 - a^3)]x \\
&\qquad + [c(prbc^3 + psac^3 + ab) + (br - as)(b^3 - a^3)]y,
\end{aligned}$$

*and $a, b, c$ are as in Lemma 2.*

As in Lemma 2, $R$ in Lemma 4 may be reduced numerically when $x, y$ are integers. Write

$$(2.13) \qquad \begin{aligned} \mu &\equiv h(prbc^3 + psac^3 + ab) + k(b^3 - a^3) \\ \sigma &\equiv c(prbc^3 + psac^3 + ab) + (br - as)(b^3 - a^3); \end{aligned}$$

$\mu, \sigma$ are integers, and in Lemma 4, $R = \mu x + \sigma y$.

**3. Consequences of Lemmas 3, 4.** In Lemma 3 take $w = 0$. Then

THEOREM 1. *With $M$ as in Lemma 3, every $\pm p\rho M^2$ (integer) is of the form $\alpha^3 + \beta^3 + p\gamma^3$, with $\alpha, \beta, \gamma$ integers, and with at most 3 exceptions $M \neq 0$, all of $\alpha, \beta, \gamma$ may be chosen $\neq 0$.*

In Lemma 4 take $w = 0$; $y = 0$. Then

THEOREM 2. *Every $\pm pg^2\sigma x^2 y$, and every $\pm 3abcpg^2x^2w$ is of the form*

$$X^3 + Y^3 + p(Z^3 + g^2\mu U^3),$$

*with $\sigma, \mu$ as in (2.13), $g$ as in (2.9). If $x, y, w$ are integers, $X, Y, Z, U$ may be chosen integers, and in each case with at most 4 exceptions, all different from zero.*

**4. Further consequences of (2.3).** Returning to Lemma 1, we now find common solutions $(x, y, z)$ of (2.3) and

$$(4.1) \qquad bx - ay = hpc.$$

Assuming (without loss of generality) as before that $(a, b) = 1$, we can find $f, g$ such that

$$bf - ag = 1. \tag{4.2}$$

Then the solution of (4.1) is

$$(x, y) = (hpcf + ka, hpcg + kb),$$

and this will give $z$ an integer in (2.3) provided

$$pc^2 \mid (hpcf + ka), \qquad pc^2 \mid (hpcg + kb).$$

Hence

$$p \mid k, \quad k = tp; \qquad c \mid h, \quad c^2 \mid t; \qquad h = mc, \quad t = nc^2,$$

and a common solution of (2.3), (4.1) is

$$\begin{aligned} x &= pc^2(an + fv), \qquad y = pc^2(bn + gv), \\ z &= pc^3 n - (a^2 f + b^2 g)v. \end{aligned} \tag{4.3}$$

Replacing $w$ by $w - pc^2 n$ in (2.4) and reducing the result we find

LEMMA 5. *If $(a, b, c)$ is any solution of (2.1) with $(a, b) = 1$, and $f, g$ are determined by (4.2), then, identically in $w, v$,*

$$\begin{aligned} (aw + pc^2 fv)^3 + (bw + pc^2 gv)^3 + p[cw - (a^2 f + b^2 g)v]^3 \\ \equiv - pv^2[3abcw - \{a^3 - (3ag + 1)pc^3\}v]. \end{aligned} \tag{4.4}$$

For numerical reductions of (4.4) we write

$$\begin{aligned} A \equiv 3abc, \qquad B \equiv (3ag + 1)pc^3 - a^3, \\ |B| = Q|A| + R, \qquad 0 \leqq Q, \qquad 0 \leqq R < |A|; \\ \operatorname{sgn}(AB) \equiv \epsilon, \qquad \operatorname{sgn} B \equiv \eta. \end{aligned} \tag{4.5}$$

LEMMA 6. *With the notations of Lemma 5 and (4.5),*

$$(aw + \alpha v)^3 + (bw + \beta v)^3 + p(cw + \gamma v)^3 \equiv - pv^2(Aw + \eta Rv).$$

*identically in $w, v$, where*

$$\alpha \equiv pc^2 f - \epsilon aQ, \qquad \beta \equiv pc^2 g - \epsilon bQ, \qquad -\gamma \equiv a^2 f + b^2 g + \epsilon cQ.$$

Since every integer $n$ is of the form $rs^2$ in at least one way we have

THEOREM 3. *Every $3abcpn$ is of the form*

$$\alpha^3 + \beta^3 + p(\gamma^3 + \eta R\delta^3)$$

*with the notation as in Lemma 6; and with at most 3 exceptions $n$, all the integers $\alpha, \beta, \gamma, \delta$ may be chosen $\neq 0$.*

THEOREM 4. *In the statement of Theorerm 2, $x^2y$ and $x^2w$ may be replaced by $n$.*

**5. Identities with fourth powers.** By integrating the identities in the preceding lemmas with respect to the parameters between suitable limits we ascend from identities involving cubes to others involving fourth powers. It will be sufficient to illustrate the general process for Lemma 6; the actual ascent in numerical examples is most readily made directly from the examples. Integrating with respect to $w$ between the limits 0 and $w$ we find

LEMMA 7. *Identically in $w$, $v$,*

$$bc(aw+\alpha v)^4+ca(bw+\beta v)^4+pab(cw+\gamma v)^4 - (bc\alpha^4+ca\beta^4+pab\gamma^4)v^4 \equiv -2pv^2w(Aw+2\eta Rv),$$

*the notation being as in Lemma 6.*

Integration with respect to $v$ between the limits 0 and $v$ gives

LEMMA 8. *Identically in $w$, $v$,*

$$\begin{aligned} 3[\beta\gamma(aw+\alpha v)^4+\gamma\alpha(bw+\beta v)^4+p\alpha\beta(cw+\gamma v)^4 \\ -(\beta\gamma a^4+\gamma\alpha b^4+p\alpha\beta c^4)w^4] \\ \equiv -pv^3(4Aw+3\eta Rv), \end{aligned}$$

*the notation being as in Lemma 6.*

For $w=1$ or $v=1$ the last two give theorems for fourth powers similar to those for cubes.

**5. Numerical examples.** An indefinite number of special results are furnished by the preceding lemmas and theorems for particular solutions of (2.1). It will suffice to illustrate Lemma 6 (a further, more systematic solution from the numerous results on hand will be given on another occasion). The obvious solution $(a,b,c,p)=(a,b,-1,a^3+b^3)$ gives some interesting results for various $a$, $b$.

The choice $(a,b,c,p)=(1,1,-1,2)$, $(f,g)=(1,0)$, gives $A=-3$, $B=-3$, $Q=1$, $R=0$, $\epsilon=1$, $\eta=-1$, $\alpha=1$, $\beta=-1$, $\gamma=0$. Hence (by Lemma 6),

(5.1) $$(w+v)^3+(w-v)^3-2w^3=6wv^2,$$

a well known identity.

(5.2) *Every $6n$ is of the form $a^3 + b^3 - 2c^3$, and if $n > 1$, we may choose $a, b, c > 0$.*

The iteration (1.2) applied to $(a, b, c) = (1, 1, -1)$ gives

$$(a, b, c) = (3, -3, 0),$$

not a solution of (2.1) since here $pabc = 0$. For

$$(a, b, c, p) = (2, -1, -1, 7)$$

we take $(f, g) = (-3, 1)$ and find $A = 6$, $B = -57$, $Q = 9$, $R = 3$, $\epsilon = -1$, $\eta = 1$, $\alpha = -3$, $\beta = -2$, $\gamma = 2$:

$$(5.3) \quad (2w - 3v)^3 - (w + 2v)^3 - 7(w - 2v)^3 = -21v^2(2w - v).$$

Replace $w$ by $-w$. Then

(5.4) *Every $42n$ is of the form $a^3 - b^3 + 7(c^3 - 3d^3)$, and if $n > 2$, we may take $a, b, c, d > 0$.*

Iteration as in (1.2) of $(a, b, c) = (2, -1, -1)$ gives the new solution $(a, b, c) = (4, 5, -3)$ of (1.2) with $p = 7$. For this we find $(f, g) = (1, 1)$, $A = -180$, $B = -2521$, $Q = 14$, $R = 1$, $\epsilon = 1$, $\eta = -1$, $\alpha = 7$, $\beta = -7$, $\gamma = 1$:

$$(5.5) \quad (4w + 7v)^3 + (5w - 7v)^3 - 7(3w - v)^3 \equiv 7v^2(180w + v).$$

(5.6) *Every $1260n$ is of the form $a^3 + b^3 - 7(c^3 + d^3)$, and if $n > 1$, we may take $a, b, c, d > 0$; by (5.4) every $1260n$ is also of the form*

$$a^3 - b^3 + 7(c^3 - 3d^3)$$

*with $a$, $b$, $c$, $d > 0$.*

In the same way we find the following. From $(a, b, c, p) = (2, 1, -1, 9)$,

$$(5.7) \quad (2w + 3v)^3 + (w - 3v)^3 - 9(w + v)^3 = 9v^2(6w - v);$$

from $(a, b, c, p) = (2, 3, -1, 35)$,

$$(5.8) \quad (2w + 7v)^3 + (3w - 7v)^3 - 35(w - v)^3 = 35v^2(18w + v);$$

from $(a, b, c, p) = (3, -2, -1, 19)$,

(5.9) $(2w+5v)^3-(3w-2v)^3+19(w-2v)^3=19v^2(18w-v)$;

from $(a,b,c,p)=(5,-4,-1,61)$,

(5.10) $(4w+13v)^3-(5w+v)^3+61(w-3v)^3=183v^2(20w+3v)$.

From (5.7)-(5.10) we write down the results corresponding to (5.6), etc. Thus from (5.10),

(5.11) *Every* $3660n$ *is of the form* $a^3-b^3+61(c^3-9d^3)$, *and if* $n>3$, *we may take* $a,b,c,d>0$; *every* $183(20n+3)$ *is of the form* $a^3-b^3+61c^3$, *and if* $n>3$, $a,b,c>0$.

One example of § 4 will suffice. Integrating (5.7) with respect to $v$ between 0 and $v$ we get

(5.12) $(2w+3v)^4-(w-3v)^4+3[4w^4+9v^4-9(w+v)^4]=216wv^3$;

(5.13) *Every* $27(8n-1)$ *is of the form* $a^4-b^4+3(4c^4-9d^4)$, *with* $a,b,c,d>0$ *if* $n>3$.

We give some miscellaneous examples, illustrative of general devices. Taking $v=\pm(1,2,3)$ in

$$(20w-3v)^3-(17w-3v)^3-9(7w-v)^3=9v(6w-v)^2,$$

obtained by the preceding methods, we get

(5.14)
$$\begin{aligned}
&(20w-3)^3-(17w-3)^3-9(7w-1)^3=9(6w-1)^2,\\
&8(10w-3)^3-(17w-6)^3-9(7w-2)^3=72(3w-1)^2,\\
&(20w-9)^3-(17u-9)^3-9(7u-3)^3=243(2w-1)^2,\\
&64(5w-3)^3-(17w-12)^3-9(7w-4)^3=144(3w-2)^2,\\
&125(4w-3)^3-(17w-15)^3-9(7w-5)^3=45(6w-5)^2,\\
&8(10w-9)^3-(17w-18)^3-9(7w-6)^3=1944(w-1)^2.
\end{aligned}$$

Hence, for example, *every* $72(3n-1)^2$ *is of the form* $8a^3-b^3-9c^3$, *with* $a,b,c>0$ *if* $n>0$. An interesting specimen of this kind, from another identity, is

(5.15) *Every* $168n^2$ *is of the form* $a^3+8b^3-7c^3$, *with* $a,b,c>0$ *if* $n>0$, *and every* $21(2n+1)^2$ *is of the form* $a^3+b^3-7c^3$, *with* $a,b,c>0$ *if* $n\geqq 0$.

Another kind is illustrated from the pair

$$(2w+3v)^3+(w-3v)^3-9(w+v)^3=9v^2(6w-v),$$
$$(2w-7v)^3+(3w+7v)^3-35(w+v)^3=35v^2(18w-v).$$

In the first replace $w$ by $3w$ and subtract from the second. Then

$$35[3(2w+v)^3+3(w-v)^3-(3w+v)^3+(w+v)^3]$$
$$=(2w-7v)^3+(3w+7v)^3.$$

In this we now make any term, say $(3w+7v)^3$, equal to $x^3$. Hence (in this case) $w=-2x-7u$, $v=x+3u$, and we get

$$(5.16)\quad x^3=(11x+35u)^3+35[(5x+18u)^3$$
$$-(x+4u)^3-3(3x+10u)^3-3(3x+11u)^3];$$

(5.17) *Every* $n^3$ *is of the form* $a^3+35(b^3-c^3-3d^3-3e^3)$, *and if* $n\neq 0$, *all of* $a,\cdots,e$ *may be chosen* $>0$ *in an infinity of ways.*

Integration of (5.16) gives

$$(5.18)\quad 11x^4=(11x+35u)^4+7[11(5x+18u)^4+224u^4]$$
$$-385[(x+4u)^4+(3x+10u)^4+(3x+11u)^4];$$

and hence, on replacing $u$ by $11u$,

$$(5.19)\quad x^4=1331(x+35u)^4+7[(5x+198u)^4+42592u^4]$$
$$-35[(x+44u)^4+(3x+110u)^4+(3x+121u)^4];$$

(5.20) *Every* $n^4$ *is of the form*

$$1331a^4+7(b^4+42592c^4)-35(d^4+e^4+f^4),$$

*and all of* $a,\cdots,f$ *may be chosen* $>0$ *in an infinity of ways if* $n\neq 0$.

Differentiation of (5.16) with respect to $x$ gives

$$(5.21)\quad x^2=11(11x+35u)^2+35[5(5x+18u)^2$$
$$-(x+4u)^2-9(3x+10u)^2-9(3x+11u)^2];$$

(5.22) *Every* $n^2$ *is of the form* $11a^2+35(5b^2-c^2-9d^2-9e^2)$, *and if* $n\neq 0$, *all of* $a,\cdots,e$ *may be chosen* $>0$ *in an infinity of ways.*

Differentiating (5.16) with respect to $u$, and replacing $x$ by $16w-35v$, $u$ by $-5w+11v$ in the result, gives

$$(5.23)\quad w^2 = 4(4w-9v)^2 + 3[10(2w-5v)^2 + 11(7w-16u)^2 - 6(10w-23v)^2];$$

(5.24) *Every* $n^2$ *is of the form* $4a^2 + 3(10b^2 + 11c^2 - 6d^2)$, *and if* $n \neq 0$, *all of* $a, \cdots, d$ may be chosen $> 0$ *in an infinity of ways.*

Differentiation of the last of (5.14) gives

$$(5.25)\quad 80(10w+1)^2 - 17(17w-1)^2 - 63(7w+1)^2 = 1296w;$$

(5.26) *Every* $1296n$ *is of the form* $80a^2 - 17b^2 - 63c^2$, *with* $a, b, c > 0$.

A general result of the last type follows from Lemma 6, with the notation as there:

(5.27) *Every* $-pabcv^2$ *is of the form*

$$a(aw+\alpha v)^2 + b(bw+\beta v)^2 + pc(cw+\gamma v)^2.$$

An example of Lemma 2 with $c \neq -1$ is

$$(5.28)\quad 17(7w+107v)^3 + (w-31v)^3 - (18w+275v)^3 = 17v^3(378w-55v).$$

The substitution $w = 8x + 55y$, $v = -55x + 378y$ transforms (5.28) into

$$(14981x - 104940y)^3 + (1713x - 11663y)^3 - 17(5829x - 40831y)^3 = 17x(55x - 378y)^2;$$

hence *every* $17(378n-55)^2$ *is of the form* $a^3 + b^3 - 17c^3$, *and if* $n > 7$, *all of* $a, b, c$ *may be chosen* $> 0$.

**6. Second method.** This can be applied to any number of terms, here illustrated for 3. Let $a, \cdots, \gamma$ be such that

$$(6.1)\quad a + b + c = \alpha + \beta + \gamma = 0, \qquad abc \neq 0.$$

Then, identically in $x, y$,

$$(6.2)\quad (ax + \alpha y) + (bx + \beta y) + (cx + \gamma y) \equiv 0.$$

Two integrations of (6.2) with respect to $x$ between the limits 0, $x$ give

(6.3) $$b^2c^2(ax+\alpha y)^3 + c^2a^2(bx+\beta y)^3 + a^2b^2(cx+\gamma y)^3 \equiv y^2[3abc(bc\alpha^2+ca\beta^2+ab\gamma^2)x + (b^2c^2\alpha^3+c^2a^2\beta^3+a^2b^2\gamma^3)y]$$

and it is clear that the restriction $abc \neq 0$ in (6.1) may be suppressed. A simple reduction by (6.1) gives

$$bc\alpha^2+ca\beta^2+ab\gamma^2 = -(b\alpha-a\beta)^2,$$
$$b^2c^2\alpha^3+c^2a^2\beta^3+a^2b^2\gamma^3 = (b\alpha-a\beta)^2[a(a+2b)\beta+b(2a+b)\alpha],$$

and we have

LEMMA 9. *Identically in $x$, $y$,*

$$b^2c^2(ax+\alpha y)^3 + c^2a^2(bx+\beta y)^3 + a^2b^2(cx+\gamma y)^3 \equiv -(b\alpha-a\beta)^2y^2[3abcx - \{b(2a+b)\alpha + a(a+2b)\beta\}y],$$

*where $a,\cdots,\gamma$ are such that*

$$a+b+c = \alpha+\beta+\gamma = 0.$$

If $(a,b)=d$, and $(a,b,c)=d(a_1,b_1,c_1)$, the identity resulting from the last has $(a,b,c,x)$ replaced by $(a_1,b_1,c_1,dx)$, or, dropping suffixes, we recover the preceding identity with $(a,b,c)=1$ and $x$ replaced by $dx$. Hence there is no loss in generality in assuming $(a,b)=1$ in Lemma 9. We can therefore choose $f$, $g$ as in (4.2), and get as the solution of $b\alpha-a\beta=u$,

(6.4) $$\alpha = fu+av, \qquad \beta = gu+bv.$$

Hence, from Lemma 9, follows

LEMMA 10. *If, without loss of generality, $(a,b)=1$, and $f$, $g$ are such that $bf-ag=1$, then, identically in $w$, $z$,*

$$(a+b)^2[b^2(aw+fz)^3+a^2(bw+gz)^3] - a^2b^2[(a+b)w+(f+g)z]^3 \equiv z^2[3ab(a+b)w + \{(a+2b)ag+(2a+b)bf\}z].$$

In obtaining this, the following change of notation was made, $x+vy=w$, $uy=z$. We give a few of the simplest examples. For

$$(a,b,f,g) = (3,-2,1,-1)$$

we get

(6.4) $$9(2w+z)^3+36w^3-4(3w+z)^3 = z^2(18w+5z);$$

(6.5) *Every* $18n+5$ *is of the form* $9a^3+36b^3-4c^3$, *with all of* $a, b, c>0$ *if* $n>0$.

From $(a, b, f, g) = (4, -3, 1, -1)$,

(6.6) $16(3w+1)^3+144w^3-9(4w+1)^3=36w+7$;

(6.7) *Every* $36n+7$ *is of the form* $16a^3+144b^3-9c^3$, *with all of* $a, b, c>0$ *if* $n>0$.

(6.8) *If* $(a, b)=1$, $bf-ag=1$, *every*

$$3ab(a+b)n+\{(a+2b)ag+(2a+b)bf\}$$

*is of the form*

$$a^2(a+b)^2A^3+b^2(a+b)^2B^3-a^2b^2C^3,$$

*and with at most 3 exceptions* $n$, *all of* $A, B, C$ *may be chosen* $>0$.

Differentiating the identity in Lemma 10 with respect to $w$ we get

(6.9) *Every* $n^2$ *is of the form*

$$(a+b)(aA^2+bB^2)-abC^2,$$

*where* $(a, b)=1$, *and all the integers* $A, B, C$ *may be chosen* $\neq 0$ *in an infinity of ways.*

We have

$$A=bm+gn, \qquad B=am+fn, \qquad C=(a+b)m+(f+g)n,$$

where $m$ is an arbitrary integer, and $f, g$ are as in Lemma 10.

**7. Simultaneous solutions of (2.1).** Let $(a, b, c)(\alpha, \beta, \gamma)$ be two distinct solutions of (2.1). Then, identically in $x, y$,

$$(ax+\alpha y)^3+(bx+\beta y)^3+p(cx+\gamma y)^3 \equiv 3xy(Px+Qy),$$
$$P \equiv a^2\alpha+b^2\beta+pc^2\gamma, \qquad Q \equiv a\alpha^2+b\beta^2+pc\gamma^2.$$

Write

(7.1) $A \equiv a^3+2b^3,\ B \equiv 2a^3+b^3,\ C \equiv a^3-b^3,\ D \equiv a^6+a^3b^3+b^6,$

so that $(\alpha, \beta, \gamma)=(aA, -bB, cC)$ is the first iterate of $(a, b, c)$ obtained by (1.2). A short reduction gives

(7.2) $P=0, \qquad Q=-9pa^3b^3c^3$;

(7.3) $a^3(x+Ay)^3+b^3(x-By)^3+pc^3(x+Cy)^3=-27pa^3b^3c^3xy^2.$

(7.4) *If* $(a, b, c)$ *is any solution of* (2.1), *every* $-27pa^3b^3c^3n$, *and every* $-27pa^3b^3c^3n^2$, *is of the form* $a^3e^3 + b^3f^3 + pc^3g^3$, *and with at most* 3 *exceptions* $n$, *in each case*, $efg \neq 0$.

In (7.3) we take $x = y$ and find

(7.5) *A two-fold infinity of solutions of*

$$x^3 + y^3 = (u^3 + v^3)(z^3 + w^3)$$

*in integers* $x, y, z, w, u, v$ *is*

$$x = a(1 + a^3 + 2b^3), \qquad u = a, \qquad z = 1 + a^3 - b^3,$$
$$y = b(1 - 2a^3 - b^3), \qquad v = b, \qquad w = 3ab,$$

*where* $a, b$ *are arbitrary integers.*

Integrating (7.3) with respect to $x$ from 0 to $x$, and reducing the constant of integration, we find (see (7.1)),

(7.6) $a^3A^4 + b^3B^4 + pc^3C^4 = 27pa^3b^3c^3D$;

(7.7) $$a^3(x + Ay)^4 + b^3(x - By)^4 + pc^3(x + Cy)^4 + 27pa^3b^3c^3Dy^4 = -54a^3b^3c^3x^2y^2;$$

(7.8) *If* $(a, b, c)$ *is any solution of* (2.1), *every* $54pa^3b^3c^3n^2$ *is of the form*

$$27pa^3b^3c^3D\delta^4 - a^3\alpha^4 - b^3\beta^4 - c^3\gamma^4, \quad \textit{with} \quad \delta, \alpha, \beta, \gamma > 0$$

*if* $n$ *is different from* $-a^3 - 2b^3$, $2a^3 + b^3$, $b^3 - a^3$.

Integrating (7.7) with respect to $y$ from 0 to $y$, and reducing as before, we find

(7.9) $a^3BC - b^3CA + pc^3AB = 9pa^3b^3c^3$;

(7.10) $$a^3BC(x + Ay)^5 - b^3CA(x - By)^5 + pc^3AB(x + Cy)^5 + 27pa^3b^3c^3ABCDy^5 - 9pa^3b^3c^3x^5 = -90pa^3b^3c^3ABCx^2y^3;$$

(7.11) *Every* $-90pa^3b^3c^3ABCn^2$, *and every* $90pa^3b^3c^3ABCn^3$, *is of the form* $a^3BC\alpha^5 - b^3CA\beta^5 + pc^3AB\gamma^5 + 9pa^3b^3c^3(3ABCD\delta^5 - \epsilon^5)$, *the notation being as in* (2.1), (7.1), *with* $\alpha\beta\gamma\delta\epsilon \neq 0$ *if* $n \neq -A, B, -C$ *for the first, and* $\alpha\beta\gamma\delta\epsilon \neq 0$ *for the second.*

Integration of (7.10) with respect to $x$ between 0 and $x$, and reduction of the constant of integration, gives

$$(7.12)\qquad a^3A^5 - b^3B^5 + pc^3C^5 = 9pa^3b^3c^3ABC;$$

$$(7.13)\qquad \begin{aligned} -a^3BC(x+Ay)^6 + b^3CA(x-By)^6 - pc^3AB(x+Cy)^6 \\ + 3pa^3b^3c^3(3x^3 - ABCy^3)(x^3 - 3ABCy^3) \\ = 162pa^3b^3c^3ABCDxy^5; \end{aligned}$$

(7.14) *Every* $162pa^3b^3c^3ABCDn$ *is of the form*

$$3pa^3b^3c^3(3\delta^3 - ABC\epsilon^3)(\delta^3 - 3ABC\epsilon^3) - a^3BC\alpha^6 + b^3CA\beta^6 - pc^3AB\gamma^6,$$

*the notation being as in* (2.1), (7.1), *and with at most* 3 *exceptions* $n$, *all of* $\alpha, \cdots, \epsilon$ *may be chosen* $> 0$.

The processes of this section can obviously be continued indefinitely. From (7.12) we note that

(7.15) *For an infinity of integers* $p$,

$$x^3u^5 + y^3v^5 - pz^3w^5 = 9px^3y^3z^3uvw$$

*is solvable in integers* $x, \cdots$ *with* $xyzuvw \neq 0$.

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# ON THE NORMAL FORMS OF LINEAR CANONICAL TRANSFORMATIONS IN DYNAMICS.

By John Williamson.

Let $n$ be the number of degrees of freedom of a linear conservative dynamical system and let the point $(q_1, q_2, \cdots, q_n;\ p_1, p_2, \cdots, p_n)$ of the phase space be denoted by $x = (x_1, x_2, \cdots, x_{2n})$. A system of $2n$ ordinary differential equations of the first order, which are homogeneous, linear and do not contain $t$ explicitly, is a canonical system, if, and only if, the differential equations can be written in the form

$$G\frac{dx}{dt} = Hx,$$

where $H$ is a real symmetric matrix of order $2n$ and $G$ is the skew symmetric matrix $\begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}$, and $E$ the unit matrix of order $n$. A non-singular linear transformation

$$y = Ax$$

is said to be a canonical transformation, if it transforms every linear canonical system into a linear canonical system. It is known that the transformation of matrix $A$ is canonical if, and only if,

$$\text{(i)} \qquad A'GA = sG$$

where $s$ is a constant.[1] It can be assumed without loss of generality that[2] $s = 1$ and accordingly we shall call a matrix $A$ a canonical matrix if it satisfies (i) with $s = +1$.

In a previous paper[3] normal forms for dynamical systems under canonical transformations were found and here we determine normal forms for canonical matrices under canonical transformations. These normal forms are not completely determined by the elementary divisors of the canonical matrix, so that two canonical matrices, which are similar, are not necessarily similar under a canonical transformation.

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[1] A. Wintner, "On the linear conservative dynamical systems," *Annali di matematica pura ed applicata*, ser. 4, tomo 13 (1934-35), pp. 105-112.

[2] E. R. van Kampen and A. Wintner, "On the canonical transformations of Hamiltonian systems," *American Journal of Mathematics*, vol. 58 (1936), pp. 851-863.

[3] John Williamson, "On the algebraic problem concerning the normal forms of linear dynamical systems," *American Journal of Mathematics*, vol. 58 (1936), pp. 141-163.

When considering the solutions of the equations of variation belonging to a periodic solution of conservative non-linear dynamical systems, the question of the occurrence of secular terms is known to depend on the elementary divisors of a canonical matrix.[4] In fact the degree of the highest secular term occurring is determined by the greatest exponent of the elementary divisors. For this reason, it is of interest, that it is possible for a canonical matrix to have an elementary divisor of order $2m$ (§ 6, Result $\mathrm{III}_a$).[5]

In the following sections the problem is considered from a purely algebraic point of view and in section 1 is reduced to a simpler one of a similar nature; sections 2 and 3 are devoted to the proofs of preliminary lemmas, while the main results are obtained in the remaining sections.

**1. Simplification of the problem.** Let $E$ be the unit matrix of order $n$ and $G$ the skew-symmetric matrix $G = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}$ of order $2n$. A real matrix $A_i$ is said to be a *canonical matrix*, if

$$(1) \qquad A_i G A'_i = G,$$

where $A'_i$ is the transposed of $A_i$. We shall be interested in determining necessary and sufficient conditions that two canonical matrices $A_1$ and $A_2$ be similar under a *canonical transformation*; in other words that there exist a third canonical matrix $A_3$, such that

$$(2) \qquad A_2 = A_3 A_1 A_3^{-1}.$$

We first reduce this problem to a somewhat simpler one.

If $A_1$ and $A_2$ are two canonical matrices, which are similar, and a matrix $Q$, to be specified later, is similar to $A_1$, then $Q$ is similar to $A_2$. There exist, therefore, two non-singular matrices $R_1$ and $R_2$, such that

$$(3) \qquad R_i A_i R_i^{-1} = Q \qquad (i = 1, 2).$$

The matrices

$$(4) \qquad R_i G R'_i = S_i \qquad (i = 1, 2),$$

are skew symmetric and are left invariant by $Q$, that is, satisfy the equations,

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[4] A. Wintner, "Three notes on characteristic exponents and equations of variation in celestial mechanics," *American Journal of Mathematics*, vol. 53 (1931), pp. 605-625.

[5] This result could be deduced by suitable modifications from papers by Alfred Loewy, "Allgemeine bilineare Formen mit konjugirt imaginären Variabeln," *Abhandlungen der Kaiserlichen Leopoldinisch-Carolinischen Deutschen Akademie der Naturforscher*, Band 71. S. S. 378-446, Halle (1898), and T. J. I'A. Bromwich, "Canonical reduction of bilinear forms," *Proceedings of the London Mathematical Society*, vol. 32 (1900), pp. 321-332.

$$QS_iQ' = S_i \qquad (i = 1, 2). \tag{5}$$

Thus, if $Q$ is any matrix similar to the two canonical matrices $A_1$ and $A_2$, there is associated with $A_1$ a skew symmetric matrix $S_1$ and with $A_2$ a skew symmetric matrix $S_2$, both of which are left invariant by $Q$.

We now prove

THEOREM 1. *A necessary and sufficient condition, that $A_1$ be similar to $A_2$ under a canonical transformation, is that there exist a non-singular matrix $H$, such that*

$$HQ = QH, \tag{6}$$

*and that the two skew symmetric matrices $S_1$ and $S_2$, associated with $A_1$ and $A_2$, satisfy*

$$HS_1H' = S_2. \tag{7}$$

*Proof.* Let a matrix $H$ satisfying (6) and (7) exist. Then

$$\begin{aligned} A_2 &= R_2^{-1}QR_2 = R_2^{-1}HQH^{-1}R_2 \text{ by (3) and (6)}, \\ &= R_2^{-1}HR_1A_1R_1^{-1}H^{-1}R_2 \text{ by (3)}, \\ &= A_3A_1A_3^{-1}, \end{aligned}$$

where $A_3 = R_2^{-1}HR_1$. Further

$$\begin{aligned} A_3GA'_3 &= R_2^{-1}HR_1GR'_1H'(R_2^{-1})' = R_2^{-1}HS_1H'(R_2^{-1})' \text{ by (4)}, \\ &= R_2^{-1}S_2(R_2^{-1})' = G \text{ by (7) and (4)}. \end{aligned}$$

Hence $A_3$ is a canonical matrix. Conversely, if (2) is satisfied and $A_3$ is a canonical matrix, the matrix $H = R_2A_3R_1^{-1}$ satisfies (6) and (7); for

$$HQH^{-1} = R_2A_3R_1^{-1}QR_1A_3^{-1}R_2^{-1} = R_2A_3A_1A_3^{-1}R_2^{-1} = R_2A_2R_2^{-1} = Q \quad \text{and}$$
$$HS_1H' = R_2A_3R_1^{-1}S_1(R_1^{-1})'A'_3R'_2 = R_2A_3GA'_3R'_2 = R_2GR'_2 = S_2.$$

Since, in the above, $Q$ is any matrix similar to $A_1$, we are at liberty to choose $Q$ in a suitable normal form. Then, if $S$ is any real skew symmetric matrix satisfying the equation

$$QSQ' = S, \tag{8}$$

we shall determine a normal form for $S$ under transformations by matrices permutable with $Q$. If $HQ = QH$ and $HSH' = S_1$, we shall call the transformation by the matrix $H$ an *admissible transformation* and shall write $S \approx S_1$.

**2. Preliminary lemmas.** When $R$ is a square matrix of order $m$, we may consider $R$ as a matrix of matrices and write

$$R = (R_{ij}) \qquad (i, j = 1, 2, \cdots, t), \tag{9}$$

where $R_{ij}$ is a matrix of $r_i$ rows and $r_j$ columns and $r_1 + r_2 + \cdots + r_t = m$. If $S$ is a second $m$-rowed square matrix and $S$ is written as a matrix of matrices

$$S = (S_{ij}) \qquad (i, j = 1, 2, \cdots, t), \tag{10}$$

where $S_{ij}$ is also a matrix of $r_i$ rows and $r_j$ columns, we shall say that $R$ and $S$ are *similarly partitioned* or that (10) is a partition of $S$ similar to that of $R$ in (9). If in (9), when $i$ is different from $j$, $R_{ij}$ is the zero matrix, we shall call $R$ a *diagonal block* matrix and write

$$R = [R_{11}, R_{22}, \cdots, R_{tt}].$$

LEMMA 1. *If the matrices $S_1$, $S_2$ and $Q$ satisfy* (5), *then $S_2 = MS_1$, where $MQ = QM$.*

*Proof.* Since $S_1$ and $S_2$ are non-singular, $Q$ is non-singular and accordingly

$$(Q')^{-1} = S_1^{-1}QS_1 = S_2^{-1}QS_2,$$

so that

$$S_2S_1^{-1}Q = QS_2S_1^{-1}.$$

If $M = S_2S_1^{-1}$, then $MQ = QM$ and $S_2 = MS_1$.

LEMMA 2. *If $Q = [Q_1, Q_2]$ and no latent root of $Q_1$ is the reciprocal of a latent root of $Q_2$, a matrix $S$, which satisfies* (8), *is of the form $[S_{11}, S_{22}]$ and*

$$Q_iS_{ii}Q'_i = S_{ii} \qquad (i = 1, 2).$$

*Proof.* Let

$$S = (S_{ij}) \qquad (i, j = 1, 2),$$

be a partition of $S$ similar to that of $Q$. Then

$$Q_iS_{ij} = S_{ij}(Q'_j)^{-1} \qquad (i, j = 1, 2). \tag{11}$$

Since, by hypothesis, no latent root of $Q_1$ is the reciprocal of a latent root of $Q_2$, no latent root of $Q_1$ is the same as a latent root of $(Q'_2)^{-1}$. Therefore, as a consequence of (11), $S_{12} = 0$. Similarly $S_{21} = 0$ and the lemma is proved.

LEMMA 3. *Let $Q = [Q_1, Q_2]$ and let $S$ be a skew-symmetric matrix satisfying* (8). *If $S = (S_{ij})$ $(i, j = 1, 2)$, is a partition of $S$ similar to that of $Q$ and, if $S_{11}$ is non-singular, then*

$$S \approx S_1,$$

*where $S_1 = [S_{11}, T_{22}]$.*

*Proof.* Let $E_i$ be the unit matrix of the same order as $Q_i$ and $H$ be the matrix

$$H=\begin{pmatrix} E_1 & 0 \\ -S_{21}S_{11}^{-1} & E_2 \end{pmatrix}.$$

As a consequence of (11),

$$S_{21}S_{11}^{-1}Q_1=S_{21}(Q'_1)^{-1}S_{11}^{-1}=Q_2S_{21}S_{11}^{-1}.$$

Hence $HQ=QH$. Since $S$ is skew-symmetric,

$$HSH'=\begin{pmatrix} E_1 & 0 \\ -S_{21}S_{11}^{-1} & E_2 \end{pmatrix}\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}\begin{pmatrix} E_1 & -S_{11}^{-1}S_{12} \\ 0 & E_2 \end{pmatrix}=[S_{11},T_{22}],$$

where $T_{22}=S_{22}-S_{21}S_{11}^{-1}S_{12}$.

**3. Normal form of $Q$.** Let $p$ be a real number or else the two rowed real matrix

$$p=\begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \text{ where } b\neq 0.$$

Let $E_i$ denote the unit matrix of order $e_i$ and $U_i$ the auxiliary unit matrix of the same order.[6] The matrix,

$$P_i=pE_i+pU_i, \tag{12}$$

has the single elementary divisor $(\lambda-p)^{e_i}$ or the two elementary divisors $(\lambda-a+ib)^{e_i}$, $(\lambda-a-ib)^{e_i}$, according as $p$ is a matrix of order one or two.[7] The diagonal block matrix,

$$\pi=[P_1,P_2,\cdots,P_t], \tag{13}$$

has therefore the elementary divisors $(\lambda-p)^{e_j}$ or $(\lambda-a+ib)^{e_j}$, $(\lambda-a-ib)^{e_j}$ $(j=1,2,\cdots,t)$. We may take $Q$ in the normal form,

$$Q=[\pi_1,\pi_2,\cdots,\pi_k], \tag{14}$$

where the matrix $\pi_j$ is obtained from $\pi$ in (13) by writing $p_j$ for $p$, $e_{ij}$ for $e_i$, and $t_j$ for $t$. Further $p_j\neq p_i$, if $j$ is different from $i$.

If $H$ is a matrix commutative with $Q$, $H$ is a diagonal block matrix $[H_1,H_2,\cdots,H_k]$, where

$$H_j\pi_j=\pi_jH_j \qquad (j=1,2,\cdots,k). \tag{15}$$

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[6] Cf. Turnbull and Aitken, *Canonical Matrices*, p. 62.

[7] By the elementary divisors of a matrix $A$ we mean the elementary divisors of $A-\lambda E$.

[8] John Williamson, "The idempotent and nilpotent elements of a matrix," *American Journal of Mathematics*, vol. 58 (1936), p. 477.

But the form of a matrix $H_j$ satisfying (15) is known.[8] In fact, if $W\pi = \pi W$ and $W = (W_{ij})$ $(i, j = 1, 2, \cdots, t)$, is a partition of $W$ similar to that of $\pi$ in (13) and, if $e_i \geqq e_j$, then

$$W_{ij} = \begin{pmatrix} F_{ij} \\ 0 \end{pmatrix} \quad \text{and} \quad W_{ji} = (0, F_{ji}),$$

where $F_{ij}$ and $F_{ji}$ are square matrices of order $e_j$. Moreover $F_{ij}$ and $F_{ji}$ are both polynomials in $U_j$ with coefficients, which are polynomials in $p$. More exactly

$$F_{ij} = \sum_{a=0}^{e_j-1} f_{ija}(p) U_j^a,$$

while

(16) $$F'_{ij} = \sum_{a=0}^{e_j-1} f_{ija}(p') U'^a_j.$$

If $i$ denotes the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $p$ is a two rowed matrix,

$$p = a + ib \quad \text{and} \quad p' = a - ib = \bar{p}.$$

With this notation (16) becomes

(17) $$F'_{ij} = \sum_{a=0}^{e_j-1} \bar{f}_{ija} U'^a_j.$$

Let $T_j$ be the counter unit matrix of order $e_j$. Then

(18) $$T_j U_j = U'_j T_j$$

and as a consequence of (17)

(19) $$T_j F'_{ij} = \bar{F}_{ij} T_j.$$

LEMMA 4. *Let $T_j W'_{ij} = \tilde{W}_{ij} T_i$. If $e_i = e_j$, $\tilde{W}_{ij} = \bar{W}_{ij}$. If $e_i > e_j$, the element in the first row and first column of $\tilde{W}_{ij}$ is zero.*

*Proof.* Let $e_i \geqq e_j$. Then

$$\begin{aligned} T_j W'_{ij} &= (T_j F'_{ij}, 0) = (\bar{F}_{ij} T_j, 0) \text{ by (19)}, \\ &= (0, \bar{F}_{ij}) T_i. \end{aligned}$$

Hence $\tilde{W}_{ij} = (0, \bar{F}_{ij})$ and the lemma is proved.

**4. Reduction of $S$.** Let

(20) $$Q = [Q_1, Q_2, \cdots, Q_k],$$

where no latent root of $Q_1$ has absolute value 1, each latent of $Q_2$ is equal to 1, each latent root of $Q_3$ is equal to $-1$ and each latent roof of $Q_j$, $j > 3$, is equal to $a_j \pm ib_j$, where $a_j^2 + b_j^2 = 1$ and $a_j + ib_j \neq a_r + ib_r$ unless $r = j$. Then,

if $S$ is a skew-symmetric matrix satisfying (8), as a consequence of Lemma 2, $S = [S_1, S_2, \cdots, S_k]$, where

$$Q_j S_j Q'_j = S_j, \qquad (j = 1, 2, \cdots, k). \tag{21}$$

Since any matrix $H$, commutative with the matrix $Q$ in (20), is also a diagonal block matrix, we may consider each of the equations (21) separately. Since $S_1$ is non-singular, $Q_1$ is similar to $(Q'_1)^{-1}$ and, since no latent root of $Q_1$ has absolute value 1, $Q_1$ is similar to a matrix $[F_1, (F'_1)^{-1}]$, where the order of $F_1$ is one-half that of $Q_1$. Hence $Q_1$ may be replaced by the matrix $[F_1, (F'_1)^{-1}]$. It is now a consequence of (21) that

$$S_1 = \begin{pmatrix} 0 & \sigma \\ -\sigma & 0 \end{pmatrix}, \quad \text{where} \quad \sigma F_1 = F_1 \sigma.$$

If $H_1 = \begin{pmatrix} \sigma^{-1} & 0 \\ 0 & I_1 \end{pmatrix}$, where $I_1$ is the unit matrix of the same order as $F_1$,

$$H_1 [F_1, (F'_1)^{-1}] H_1^{-1} = [F_1, (F'_1)^{-1}] \quad \text{and} \quad H_1 S_1 H'_1 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} = G_1.$$

Hence we have

*Result I.* The matrix $Q_1$ may be taken in the form $Z_1 = [F_1, (F'_1)^{-1}]$. With this value of $Q_1$, $S_1 \approx G_1$.

The matrix $F_1$ is not unique and may be replaced by any matrix similar to it; in fact $F_1$ may be taken in the normal form $[\pi_1, \pi_2, \cdots, \pi_f]$, where $\pi_j$ is defined by (13) and $| p_j | \neq 1$. As a consequence of the above and Theorem 1 we have

THEOREM 2. *If $A_1$ is a canonical matrix similar to a second canonical matrix $A_2$ and, if no latent root of $A_1$ is of absolute value* 1, *$A_1$ is similar to $A_2$ under a canonical transformation.*

We next consider equations (21), when $j \geqq 2$, and for simplicity of notation temporarily drop the suffix $j$. The matrix $Q = Q_j$ is therefore of the form

$$Q = [P_1, P_2, \cdots, P_t], \qquad e_1 \geqq e_2 \geqq \cdots \geqq e_t, \tag{22}$$

where $P_i$ is defined by (12) with the added restriction that $| p | = 1$. Hence $p$ is a real orthogonal matrix of order one or two. If

$$S = (S_{ij}), \qquad (i, j = 1, 2, \cdots, t),$$

is a partition of $S$ similar to that of $Q$ in (22), equation (18) implies

$$P_i S_{ij} P'_j = S_{ij}, \qquad (i, j = 1, 2, \cdots, t),$$

or, if $S_{ij} = \sigma$,

$$P_i \sigma P'_j = \sigma. \tag{23}$$

The matrix $\sigma = (\sigma_{rs})$ in (23) is a matrix of $m = e_i$ rows and $n = e_j$ columns. On equating corresponding elements in (23) we obtain

$$(24)\quad p(\sigma_{rs} + \sigma_{r+1,s} + \sigma_{r,s+1} + \sigma_{r+1,s+1})p' = \sigma_{rs},\ (r = 1, 2, \cdots, m;\, s = 1, 2, \cdots, n),$$

with the understanding that $\sigma_{m+1,s} = \sigma_{r,n+1} = 0$. If $p = \pm 1$, (24) reduces to

$$(25)\qquad \sigma_{r+1,s} + \sigma_{r,s+1} + \sigma_{r+1,s+1} = 0.$$

On substituting $s = n,\ n-1,\ n-2, \cdots$, successively in (25) we have

$$(26)\quad \sigma_{r+1,n} = \sigma_{r+2,n-1} = \cdots = \sigma_{r+s+1,n-s} = 0,\ \ (r = 1, 2, \cdots, m;\ s = 1, 2, \cdots, n),$$

and, on substituting $r = m, m-1, m-2, \cdots$, successively,

$$(27)\quad \sigma_{m,s+1} = \sigma_{m-1,s+2} = \cdots = \sigma_{m-r,r+s+1} = 0,\ (r = 1, 2, \cdots, m;\, s = 1, 2, \cdots, n).$$

We easily deduce from (25), (26) and (27),

LEMMA 5. *If $\sigma$ is a matrix satisfying (23) and, if $e_i \neq e_j$, the last row and the last column of $\sigma$ are zero. If $e_i = e_j = n$, then $\sigma_{rs} = 0$, when $r + s > n + 1$ and*

$$(28)\qquad \sigma_{n1} = -\sigma_{n-1,2} = \sigma_{n-2,3} = \cdots = (-1)^{n-1}\sigma_{1n}.$$

If $p$ is of order 2, equations (25) may be solved to give a particular matrix $S^*$, which satisfies (22) and whose elements are two rowed scalar matrices. Any other matrix $S$, which satisfies (22) is, by Lemma 1, of the form $MS^*$, where $MQ = QM$. Since the elements of $M$ are polynomials in $p$, so are the elements of $S$. Hence

$$(29)\qquad p\sigma_{rs}p' = \sigma_{rs}pp' = \sigma_{rs},$$

since $p$ is orthogonal. Accordingly, *Lemma 5 is also true, when $p$ is a two-rowed matrix.*

Let $e_1 = e_2 = \cdots = e_c > e_{c+1}$ and let $s_{ij}$ denote the element in the first column and the last row of $S_{ij}$. Then, by Lemma 5, $S_{11}$ is singular, if and only if $s_{11}$ is zero. If $S_{jj}$ is non-singular, $1 < j \leqq c$, we may interchange $S_{jj}$ and $S_{11}$ without disturbing $Q$. If $S_{jj}$ is singular for all values of $j$, $1 \leqq j \leqq c$, then

$$(30)\qquad s_{jj} = 0, \qquad (j = 1, 2, \cdots, c).$$

Since $S$ is non-singular and since, by Lemma 5, the last row of $S_{1k}$ is zero, when $k > c$, for at least one value of $j$, $1 < j \leqq c$, $s_{1j} \neq 0$. We may therefore suppose, without any loss of generality, that $s_{12} \neq 0$.

Let $I$ be the unit matrix of order $e_3 + e_4 + \cdots + e_t$ and $H_1$ the matrix $\left[\begin{pmatrix} E_1 & E_1 \\ 0 & E_1 \end{pmatrix}, I\right]$. Then $H_1$ is commutative with $Q$ and

$$H_1SH'_1 = R = (R_{ij}) \qquad (i, j = 1, 2, \cdots, t),$$

where

$$R_{11} = S_{11} + S_{12} + S_{21} + S_{22}.$$

The element in the last row and first column of $R_{11}$ is

$$r_{11} = s_{11} + s_{12} + s_{21} + s_{22} = s_{12} + s_{21} \text{ by } (30).$$

If $p$ is a two-rowed matrix the transformation by the matrix

$$H_2 = \left[\begin{pmatrix} E_1 & -iE_1 \\ 0 & E_1 \end{pmatrix}, I\right]$$

is admissible and

$$H_2SH'_2 = F = (F_{ij}), \qquad (i, j = 1, 2, \cdots, t),$$

where

$$f_{11} = i(s_{21} - s_{12}).$$

Since $S$ is skew symmetric, $S_{21} = - S'_{12}$ and by (28)

$$s_{21} = -(-1)^{e_1-1}s'_{12}.$$

Hence, if $e_1$ is even and $p$ is of order 1, $s_{21} = s'_{12} = s_{12}$ and $r_{11} = 2s_{12} \neq 0$. If $p$ is of order 2, at least one of $f_{11}$ or $r_{11}$ is different from zero and accordingly at least one of $F_{11}$ or $R_{11}$ is non-singular. Therefore, unless $e_1$ is odd and $p = \pm 1$,

$$S \approx L = (L_{ij}), \qquad (i, j = 1, 2, \cdots, t),$$

where

$$L_{11} = S_1$$

is non-singular.

If $e_1$ is odd and $p = \pm 1$, $s_{11} = - s'_{11} = 0$ and $S_{11}$ is singular. Hence $c \geqq 2$ and we may suppose that $s_{12} \neq 0$. Then the matrix

$$S_1 = (S_{ij}) \qquad (i, j = 1, 2),$$

is non-singular, since

$$|S_1| = \pm (s_{12})^{2e_1},$$

as is seen by re-arranging the rows and columns of $S_1$ in the order $1, e_1 + 1, 2, e_1 + 2$, etc. By repeated applications of Lemma 3 we therefore deduce that

$$S \approx [S_1, S_2, \cdots, S_k]. \tag{31}$$

The component matrices $S_j$ on the right of (31) are of two distinct types:

*Type a.* The matrix $S_j$ is of order $2e_j$, $p = \pm 1$, $e_j$ is odd, and $[P_j, P_j]S_j[P_j, P_j]' = S_j$.

*Type b.* The matrix $S_j$ is of order $e_j$ and $P_jS_jP'_j = S_j$.

*Reduction of type a.* For convenience we drop the suffix $j$ and write

$$[P, P]S[P, P]' = S, \quad \text{where } S = (S_{rs}), \qquad (r, s = 1, 2).$$

Hence

$$PS_{rs}P' = S_{rs} \qquad (r, s = 1, 2).$$

As a consequence of Lemma 1,

$$S_{rs} = M_{rs}X, \tag{32}$$

where $M_{rs} = M_{rs}(U)$ is a polynomial in $U = U_j$ and $X$ is a particular solution of $PXP' = X$. Since $S_{rr}$ is singular and $X$ is non-singular,

$$M_{rr}(U) = Um_{rr}(U)\,; \qquad (r = 1, 2). \tag{33}$$

If

$$\sigma_1 = \begin{pmatrix} 0 & S_{12} \\ S_{21} & 0 \end{pmatrix} \quad \text{and} \quad \sigma_2 = \begin{pmatrix} S_{11} & 0 \\ 0 & S_{22} \end{pmatrix},$$

$\sigma_1$ is non-singular and

$$\sigma_2\sigma_1^{-1} = \begin{pmatrix} 0 & M_{11}M_{21}^{-1} \\ M_{22}M_{12}^{-1} & 0 \end{pmatrix}. \tag{34}$$

The matrix $\sigma_2\sigma_1^{-1}$ is commutative with $[P, P]$ and therefore so is the matrix $H = E - \frac{1}{2}\sigma_2\sigma_1^{-1}$. Further

$$\begin{aligned} HSH' &= (E - \tfrac{1}{2}\sigma_2\sigma_1^{-1})(\sigma_1 + \sigma_2)(E - \tfrac{1}{2}\sigma_1^{-1}\sigma_2), \\ &= \tau_1 + \tau_2, \end{aligned}$$

where $\tau_1 = \sigma_1 - \frac{3}{4}\sigma_2\sigma_1^{-1}\sigma_2$ and $\tau_2 = \frac{1}{4}\sigma_2\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2$. As a consequence of (32), (33), and (34)

$$\tau_2 = [K_{11}X, K_{22}X],$$

where $K_{11}$ and $K_{22}$ are polynomials in $U$ each with a factor $U^3$, while $\tau_1$ is of the same nature as $\sigma_1$ and is non-singular. We may therefore repeat this process of reduction with $\sigma_i$ replaced by $\tau_i$ and, since $U^{e_j} = 0$, in at most $(e_j + 1)/2$ steps reduce $S$ to the form

$\begin{pmatrix} 0 & S_{12} \\ -S'_{12} & 0 \end{pmatrix}$ where $PS_{12}P' = S_{12}$.

Let $H$ be the matrix $H = [E_j, (S'_{12})^{-1}]$. Then

$$HSH' = \begin{pmatrix} 0 & E_j \\ -E_j & 0 \end{pmatrix} = G_j,$$

and

$$H[P, P]H' = [P, (S'_{12})^{-1}PS'_{12}] = [P, (P')^{-1}].$$

We have therefore

*Result II.* In type a the matrix $[P_j, P_j]$ may be replaced by

$$Z_j = [P_j, (P'_j)^{-1}].$$

Then $S_j \approx G_j$.

*Reduction of type b.* Again we drop the suffix $j$ and write $P = pE + pU$, where $P$ is of order $e$. If $T$ is the counter unit matrix of order $e$, it is a consequence of Lemma 5 that

$$S = (\sigma_0 + \sigma_1 + \cdots + \sigma_{e-1})T, \tag{37}$$

where the elements of $\sigma_k$ are all zero except in the $k$-th diagonal above the leading one. If $\sigma_{jk}$ is the non-zero element in the $j$-th row of $\sigma_k$, a simple calculation shows that

$$\sigma_{jk} = s_{k,e+1-j-k}. \tag{38}$$

The matrix $U^k\sigma_j$ is of the same type as $\sigma_{k+j}$ and, in particular, the elements of

$$\rho_k = U^k\sigma_0 \tag{39}$$

are all zero except those in the $k$-th diagonal above the leading one. The non-zero element in the $j$-th row of $\rho_k$ is

$$\rho_{jk} = \sigma_{j+k,0} = s_{j+k,e+1-j-k}. \tag{40}$$

If

$$H_k = E + qU^k, \quad \text{where} \quad qp = pq,$$

then

$$H_kS_kH'_k = C = (c_{rs}), \qquad (r, s = 1, 2, \cdots, e). \tag{41}$$

Since $TU' = UT$,

$$\begin{aligned} C &= (E + qU^k)STT(E + q'U'^k) \\ &= (E + qU^k)(\sigma_0 + \sigma_1 + \cdots \sigma_{e-1})(E + q'U^k)T \\ &= (\gamma_0 + \gamma_1 + \cdots + \gamma_{e-1})T, \end{aligned}$$

where

$$\gamma_f = \sigma_f, \qquad (f = 0, 1, 2, \cdots, k-1), \tag{42}$$

and

$$\gamma_k = \sigma_k + U^kq\sigma_0 + \sigma_0q'U^k.$$

Since, $\sigma_0U = - U\sigma_0$, this last equation becomes

$$\gamma_k = \sigma_k + (q + (-1)^kq')\sigma_0U^k.$$

The non-zero element in the $j$-th row of $\gamma_k$ is (cf. 38)

$$c_{j,e+1-j-k} = \sigma_{jk} + (q + (-1)^kq')\rho_{jk}.$$

Hence by (38) and (40),

$$(43)\qquad c_{j,e+1-j-k} = s_{j,e+1-j-k} + (q + (-1)^k q')s_{j+k,e+1-j-k}.$$

*Type* $b_1$. $e = 2m$. Let $k = e + 1 - 2j$ and $q = -s_{jj}(2s_{e+1-j,j})^{-1}$. Then, since $s_{jj}$ is skew symmetric and $s_{e+1-j,j}$ is symmetric, $q$ is skew symmetric and as a consequence of (43), $c_{jj} = 0$.

Since, by (25), $s_{j,j-1} + s_{j-1,j} + s_{jj} = 0$, if $s_{jj} = 0$, $s_{j,j-1} = -s_{j-1,j}$ and accordingly $s_{j,j-1}$ is symmetric. Therefore, when $s_{jj} = 0$, if $k = e + 2 - 2j$ and $q = -s_{j,j-1}(2s_{e+2-j,j-1})^{-1}$, it is a consequence of (43) that $c_{j,j-1} = 0$. Hence it is possible by an admissible transformation to reduce $S$ to a form, in which $s_{jj} = s_{j,j-1} = s_{j-1,j} = 0$. Equations (42) show that such a transformation does not alter the value of $s_{rs}$, when $r + s > 2j$. Therefore by giving $j$ successively the values $m, m-1, \cdots, 2, 1$ we deduce that $S \approx D$, where

$$(44)\qquad d_{11} = d_{12} = d_{21} = d_{22} = d_{23} = \cdots = d_{m,m-1} = d_{m,m} = 0.$$

Equations (44) and (25) together imply that

$$(45)\qquad d_{rs} = 0, \qquad (r, s = 1, 2, \cdots, m).$$

The non-zero elements of $D$ are now determined by means of equations (25) in the form

$$d_{rs} = d_{1e}x_{rs},$$

where $x_{rs}$ is unique. Hence

$$S = dX$$

where $X$ is uniquely determined. Since $d = d_{1e}$ is symmetric, $d$ is a scalar. Therefore the admissible transformation of matrix $E/\sqrt{d}$ reduces $dX$ to the form $\epsilon X$, where $\epsilon = \pm 1$, so that

$$S \approx \epsilon X, \qquad \epsilon = \pm 1.$$

As a consequence of (45), we have

$$(46)\qquad X = \begin{pmatrix} 0 & X_{12} \\ -X'_{12} & 0 \end{pmatrix},$$

where $X_{12}$ is a square matrix of order $m = e/2$. For example if $m = 4$,

$$X_{12} = \begin{bmatrix} 1 & 3 & 3 & 1 \\ -1 & -2 & -1 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix},$$

where, in case $p$ is a two-rowed matrix, each integer denotes the corresponding scalar matrix.[9] We have therefore

*Result III.* In type $b$, when $e_j = 2m_j$, $S_j \approx \epsilon X_j$, where $\epsilon = \pm 1$ and $X_j$ is uniquely determined.

*Type $b_2$.* $e = 2m + 1$. The matrix $p$ is necessarily a two-rowed matrix and each element $s_{rs}$ of $S$ is of the form $s_{rs} = a_{rs} + ib_{rs}$. Further, since

$$s_{e1} = (-1)^{2m} s_{1e} = -s'_{e1},$$

$s_{e1}$ is skew symmetric and, as a consequence of (28), $s_{rs}$ is skew symmetric, when $r + s = e + 1$. If $k = 1$, and $q = -a_{m,m+1}(2s_{m+1,m+1})^{-1}$, as a consequence of (43),

$$c_{m,m+1} = s_{m,m+1} - a_{m,m+1} = ib_{m,m+1}.$$

Therefore we may suppose $s_{m,m+1}$ to be skew symmetric. By a process analogous to that adopted for the case $e = 2m$ it may be shown that

$$S \approx \epsilon Y,$$

where $\epsilon = \pm 1$ and $Y$ is uniquely determined. In particular

$$y_{rs} = 0, \qquad (r, s = 1, 2, \cdots, m), \tag{47}$$

and

$$y_{rs} = 0, \qquad r + s > e + 2. \tag{48}$$

For example, if $e = 5$,

$$Y = \begin{pmatrix} 0 & 0 & i/2 & 3i/2 & i \\ 0 & 0 & -i/2 & -i & 0 \\ i/2 & -i/2 & i & 0 & 0 \\ 3i/2 & -i & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We have therefore

*Result IV.* In type $b_2$ when $e_j = 2m_j + 1$, $S_j \approx \epsilon Y_j$ where $\epsilon = \pm 1$ and $Y_j$ is uniquely determined.

By combining results I, II, III, and IV it is possible to determine a normal form for $S$ under admissible transformations. This normal form is not completely determined by the elementary divisors of $Q - \lambda E$ or of $A - \lambda E$. *With each elementary divisor of the form $(\lambda \pm 1)^{2k}$ and with each pair of conjugate elementary divisors of the form $(\lambda - a \pm ib)^k$, $a^2 + b^2 = 1$, is associated a positive or negative sign.*

[9] Cf. Turnbull and Aitken, *Canonical Matrices*, pp. 155-159.

Before proceeding to show that the elementary divisors together with the signs attached to them completely determine the normal form for $S$, we deduce

THEOREM 3. *If $A$ is a canonical matrix the determinant of $A$ has the value* 1.

This is an immediate consequence of the fact that the determinant of $A$ is the product of the latent roots of $A$ and that the latent root $-1$ must occur an even number of times (result II).

**5. Necessary conditions.** Let $A_1$ and $A_2$ be two canonical matrices, which are similar under a canonical transformation. Then, by Theorem I, the associated skew-symmetric matrices $S_1$ and $S_2$ are equivalent under an admissible transformation. The matrices $S_1$ and $S_2$ may be taken in the normal form of the previous section and are accordingly diagonal block matrices, whose component block matrices differ at most in sign. If, with the notation of (20),

$$Q = [Q_1, Q_2, \cdots, Q_k], \quad S_1 = [\sigma_1, \sigma_2, \cdots, \sigma_k] \quad \text{and} \quad S_2 = [\tau_1, \tau_2, \cdots, \tau_k],$$

there then exist $k$ non-singular matrices $W_j$ such that

$$W_j\sigma_j W'_j = \tau_j \quad \text{and} \quad W_jQ_j = Q_jW_j, \qquad (j = 1, 2, \cdots, k).$$

We need, therefore, consider only equations of the type

$$W\sigma W' = \tau, \qquad W\pi = \pi W,$$

where $\pi$ is defined by (13). If $\sigma = (\sigma_{ij})$, $\tau = (\tau_{ij})$ and $W = (W_{ij})$, $(i, j = 1, 2, \cdots, t)$, we have the equations

$$\sum_{\alpha=1}^{t} \sum_{\beta=1}^{t} W_{i\alpha}\sigma_{\alpha\beta}W'_{j\beta} = \tau_{ij}, \qquad (i, j = 1, 2, \cdots, t). \tag{49}$$

If $\sigma_{ij} = K_{ij}T_j$ and $\tau_{ij} = F_{ij}T_j$, equation (49) becomes

$$\sum_{\alpha=1}^{t} \sum_{\beta=1}^{t} W_{i\alpha}K_{\alpha\beta}T_\beta\tilde{W}'_{j\beta} = F_{ij}T_j,$$

or, by Lemma 4,

$$\sum_{\alpha=1}^{t} \sum_{\beta=1}^{t} W_{i\alpha}K_{\alpha\beta}\tilde{W}_{j\beta}T_j = F_{ij}T_j \qquad (i, j = 1, 2, \cdots, t).$$

It is a consequence of the nature of the matrices $W_{ij}$, $K_{ij}$, etc., that this last equation implies

$$\sum_{\alpha=1}^{t} \sum_{\beta=1}^{t} w_{i\alpha}k_{\alpha\beta}\tilde{w}_{j\beta} = f_{ij}, \qquad (i, j = 1, 2, \cdots, t), \tag{50}$$

where each small letter denotes the element in the first row and the first column of the matrix denoted by the corresponding capital letter. Since $\sigma$ and $\tau$ are in normal form $\sigma_{ij} = \tau_{ij} = 0$, if $e_i \neq e_j$. Further $w_{ij} = 0$, if $e_i < e_j$, and, by Lemma 4, $\bar{w}_{ij} = 0$, if $e_i > e_j$. Hence, if $e_{c-1} > e_c = e_{c+1} = \cdots = e_d > e_{d+1}$, we have as a result of (50) and Lemma 4,

$$(51) \qquad \sum_{\alpha=c}^{d} \sum_{\beta=c}^{d} w_{i\alpha} k_{\alpha\beta} \bar{w}_{j\beta} = f_{ij}, \qquad (i, j = c, c+1, \cdots, d).$$

If $B$ is the matrix whose elements are $w_{ij}$, $(i, j = c, c+1, \cdots, d)$, (50) may be written in the form

$$(52) \qquad B(k_{ij})\bar{B}' = (f_{ij}), \qquad (i, j = c, c+1, \cdots, d).$$

Since $|B|$ is a factor of $|W|$ and $W$ is non-singular so is $B$.[10] The $(d-c)$-rowed square matrices $(k_{ij})$ and $(f_{ij})$, $(i, j = c, c+1, \cdots, d)$, are therefore conjunctively equivalent. Further, as a consequence of results I-IV, $(f_{ij})$ coincides with $(k_{ij})$, unless the matrix $P_i$ is of type b. In this last case

$$(k_{ij}) = [\epsilon_c g, \epsilon_{c+1} g, \cdots, \epsilon_d g] \quad \text{and} \quad (f_{ij}) = [\epsilon'_c g, \epsilon'_{c+1} g, \cdots, \epsilon'_d g],$$

where $\epsilon_j = \pm 1$, $\epsilon'_j = \pm 1$ and $g = 1$ or $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Therefore we deduce from (52) that

$$B[\epsilon_c, \epsilon_{c+1}, \cdots, \epsilon_d]\bar{B}' = [\epsilon'_c, \epsilon'_{c+1}, \cdots, \epsilon'_d].$$

Hence the number of positive $\epsilon_j$ is the same as the number of positive $\epsilon'_j$. We may call the number of positive $\epsilon_j$ *the index of the elementary divisors* $(\lambda \pm 1)^{ec}$ *or of the pair of conjugate elementary divisors* $(\lambda - a \pm ib)^{ec}$.

Hence by Theorem 1 we have

THEOREM 4. *Necessary and sufficient conditions, that two canonical matrices $A_1$ and $A_2$ be similar under a canonical transformation, are that*

($\alpha$) *the elementary divisors of the pencil $A_1 - \lambda E$ be the same as those of the pencil $A_2 - \lambda E$, and that*

($\beta$) *the indices of all elementary divisors $(\lambda \pm 1)^{2k}$ and of all pairs of conjugate elementary divisors $(\lambda - a \pm ib)^k$, $a^2 + b^2 = 1$, be the same for both pencils.*

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[10] Cf. John Williamson, "The equivalence of non-singular pencils of hermitian matrices in an arbitrary field," *American Journal of Mathematics*, vol. 57 (1935), pp. 484-485.

**6. Normal form of a canonical matrix.** In order to determine the normal form, to which a canonical matrix $A$ may be reduced by a canonical transformation, it is only necessary, on account of Theorem 1, to reduce the associated skew symmetric matrix $S$ of section 4 to the form $G$ by a matrix $R$, and then to determine $RQR^{-1}$. As a first step we reduce each $S_j$ of type b to the form $G_j$.

*Type* $b_1$. Since $e_j = 2m$, we may write

$$P_{e_j} = P_{2m} = \begin{pmatrix} P_m & L_m \\ 0 & P_m \end{pmatrix},$$

where $P_m$ is of order $m$ and is defined by (12), while all elements of $L_m$ are zero except the element in the last row and first column which has the value $p$. By result III

$$S_j \approx \epsilon X = \epsilon \begin{pmatrix} 0 & C \\ -C' & 0 \end{pmatrix},$$

where $C$ is a non-singular matrix of order $m$. Since

$$P_{2m}\epsilon X P'_{2m} = \epsilon X,$$

it is easily verified that

$$P_m C' P'_m = C'. \tag{56}$$

Let $R = \begin{pmatrix} E & 0 \\ 0 & \epsilon(C')^{-1} \end{pmatrix}$. Then

$$R\epsilon XR' = \begin{pmatrix} 0 & E_m \\ -E_m & 0 \end{pmatrix} = G_m,$$

and

$$RP_{2m}R^{-1} = \begin{pmatrix} P_m & \epsilon L_m C' \\ 0 & (C')^{-1}P_mC' \end{pmatrix} = \begin{pmatrix} P_m & \epsilon M_m \\ 0 & (P'_m)^{-1} \end{pmatrix} \text{ by (54)},$$

where

$$M_m = \begin{bmatrix} 0 & 0 & \cdot & 0 \\ 0 & 0 & \cdot & 0 \\ \cdot & & \cdot\ \cdot & 0 \\ p & -p & \cdot & (-1)^{m-1}p \end{bmatrix}. \tag{55}$$

Accordingly we have

*Result* $III_a$. *In type* $b_1$, *when* $e_j = 2m$, $P_{e_j}$ *may be replaced by*

$$Z_{2m} = \begin{pmatrix} P_m & \epsilon M_m \\ 0 & (P'_m)^{-1} \end{pmatrix},$$

*where* $M_m$ *is defined by* (55) *and* $\epsilon = \pm 1$. *Then* $S_j \approx G_{2m}$.

For example, if $m = 3$, and $\epsilon = 1$, $P_{2m}$ may be replaced by

$$\begin{pmatrix} p & p & 0 & 0 & 0 & 0 \\ 0 & p & p & 0 & 0 & 0 \\ 0 & 0 & p & p & -p & p \\ 0 & 0 & 0 & p & 0 & 0 \\ 0 & 0 & 0 & -p & p & 0 \\ 0 & 0 & 0 & p & -p & p \end{pmatrix}.$$

If $p = 1$, this last matrix is a canonical matrix of order six with the single elementary divisor $(\lambda - 1)^6$.

*Type $b_2$. Since* $e_j = 2m + 1$ we may write

$$P_{e_j} = \begin{pmatrix} P_m & L_m \\ 0 & P_{m+1} \end{pmatrix},$$

where the only non-zero element of $L_m$ is an element $p$ in the last row and first column. By result IV

$$S_j \approx \epsilon Y,$$

and

$$Y = \begin{pmatrix} 0 & K \\ D & 0 \end{pmatrix},$$

where $D$ is a non-singular $(m+1)$-rowed matrix, while $K$ consists of the first $m$ rows of $-D'$. As in the previous case we deduce that

$$P_{m+1} D P'_{m+1} = D. \tag{56}$$

Let $R = \begin{pmatrix} E_m & 0 \\ 0 & -\epsilon D^{-1} \end{pmatrix}$. Then

$$\begin{aligned} R\epsilon Y R' &= \begin{pmatrix} E_m & 0 \\ 0 & -\epsilon D^{-1} \end{pmatrix} \begin{pmatrix} 0 & \epsilon K \\ \epsilon D & 0 \end{pmatrix} \begin{pmatrix} E_m & 0 \\ 0 & -\epsilon (D')^{-1} \end{pmatrix} \\ &= \begin{pmatrix} 0 & \epsilon K \\ -E_{m+1} & 0 \end{pmatrix} \begin{pmatrix} E_m & 0 \\ 0 & -\epsilon (D')^{-1} \end{pmatrix}. \end{aligned} \tag{57}$$

Since $K$ is formed of the first $m$ rows of $-D'$, $-K(D')^{-1} = (E_m, 0)$. Further the first $m$ elements of the last row of the product on the right of (57) are zero, while the last $m+1$ are the elements in the first row of $\epsilon(D^{-1})'$. The only element in the last column of $D^{-1}$ different from zero is the last, which has the value $(-1)^{m-1}(i)$. Therefore the only element different from zero in the first row of $(D^{-1})'$ is the last, which has the value $(-1)^{m-1}(i)' = (-1)^m i$. Accordingly it follows from (57) that

$$R\epsilon Y R' = \begin{pmatrix} 0 & E_m & 0 \\ -E_m & 0 & 0 \\ 0 & 0 & (-1)^m \epsilon i \end{pmatrix}. \tag{58}$$

But by (56)

$$(59) \qquad RP_{2m}R' = \begin{pmatrix} P_m & \epsilon N_m \\ 0 & (P'_{m+1})^{-1} \end{pmatrix} = F,$$

where the last row of $N_m$ is $p$ times the first row of $-D$, that is

$$(60) \qquad N_m = \begin{bmatrix} 0 & 0 & 0 & \cdot & 0 & 0 \\ 0 & 0 & 0 & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -\frac{ip}{2} & \frac{ip}{2} & \cdot & \cdot & (-1)^{m-2}\frac{ip}{2} & (-1)^{m-1}ip \end{bmatrix}.$$

It is not possible to proceed any further with the reduction without breaking up some of the two-rowed matrices into their component elements. Accordingly we write $N_m = K_m\alpha_1\alpha_2$, where $\alpha_1$ and $\alpha_2$ are matrices of a single column, and $(P'_{m+1})^{-1}$ in the form

$$(P'_{m+1})^{-1} = \begin{bmatrix} (P'_m)^{-1} & \gamma_1 & \gamma_2 \\ \delta_1 & a & b \\ \delta_2 & -b & a \end{bmatrix},$$

where $\gamma_1, \gamma_2$ are matrices of a single column and $\delta_1$, $\delta_2$ matrices of a single row. Then the matrix $F$ in (59) becomes

$$\begin{bmatrix} P_m & \epsilon K_m & \epsilon\alpha_1 & \epsilon\alpha_2 \\ 0 & (P'_m)^{-1} & \gamma_1 & \gamma_2 \\ 0 & \delta_1 & a & b \\ 0 & \delta_2 & -b & a \end{bmatrix}.$$

If $\epsilon = (-1)^m$, so that $\epsilon(-1)^m i = i$, a simple interchange of rows and columns reduces the matrix on the right of (58) to

$$\begin{bmatrix} 0 & 0 & E_m & 0 \\ 0 & 0 & 0 & 1 \\ -E_m & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \equiv G_{e_j}$$

and $F$ to

$$(61) \qquad Z_{e_j} = \begin{bmatrix} P_m & \epsilon\alpha_1 & \epsilon K_m & \epsilon\alpha_2 \\ 0 & a & \delta_1 & b \\ 0 & \gamma_1 & (P_m) & \gamma_2 \\ 0 & -b & \delta_2 & a \end{bmatrix}.$$

On the other hand, if $\epsilon = (-1)^{m-1}$, since $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$,

the matrix on the right of (58) may be reduced to $G_{e_j}$ and $F$ to a matrix obtained from (61) by interchanging the subscripts 1 and 2 and $b$ with $-b$. We therefore have

*Result $IV_a$.* In type $b_2$, when $e_j = 2m + 1$, $P_{2m+1}$ may be replaced by one of the forms (61). Then $S_j \approx G_{e_j}$.

By the above processes we may reduce $S$ to the form $[G_1, G_2, \cdots, G_k]$, where $G_i = \begin{pmatrix} 0 & E_i \\ -E_i & 0 \end{pmatrix}$ and $Q$ to the form $[Z_1, Z_2, \cdots, Z_k]$, where $Z_i$ is determined from one of the results I, II, $\text{III}_a$ and $\text{IV}_a$. Let

$$Z_j = \begin{pmatrix} Z_{j,11} & Z_{j,12} \\ Z_{j,21} & Z_{j,22} \end{pmatrix}, \qquad (j = 1, 2, \cdots, k),$$

where $Z_{j,rs}$ is a square matrix of the same order as $E_j$. Then by a simple interchange of rows and the same interchange of columns, the matrix $[G_1, G_2, \cdots, G_k]$, may be reduced to $G$ and at the same time $Z_1, Z_2, \cdots, Z_k$ to

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}. \tag{62}$$

The matrices $A_{rs}$ in (62) are defined by

$$A_{rs} = [Z_{1,rs}, Z_{2,rs}, \cdots, Z_{k,rs}], \qquad (r, s = 1, 2). \tag{63}$$

The matrices (62) are uniquely determined, apart from a rearrangement of rows and the same rearrangement of the columns, by the elementary divisors of $A - \lambda E$ and the indices of these elementary divisors. Therefore we have

THEOREM 5. *Any canonical matrix is similar under a canonical transformation to one and (essentially) only one of the matrices* (62).

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# CRITERIA FOR CERTAIN HIGHER CONGRUENCES.*


By Leonard Carlitz.


**1. Introduction.** The congruences in question are of the form

$$\sum_{i=0}^{s} (-1)^i \begin{bmatrix} s \\ s-i \end{bmatrix}^{p^{n(i-1)}} u^{p^{ni}} \equiv M \pmod P, \tag{1.1}$$

where $M$ and $P$ are polynomials in an indeterminate $x$ with coefficients in the Galois field $GF(p^n)$ of order $p^n$, and $P$ is irreducible. As for the coefficients in the left member of (1.1), if we put

$$[k] = x^{p^{nk}} - x \qquad (k = 0, 1, 2, \cdots), \tag{1.2}$$

$$\begin{aligned} F_k &= [k][k-1]^{p^n} \cdots [1]^{p^{n(k-1)}}, & F_0 &= 1, \\ L_k &= [k][k-1] \cdots [1], & L_0 &= 1, \end{aligned} \tag{1.3}$$

then we define

$$\begin{bmatrix} s \\ i \end{bmatrix} = \frac{F_s}{F_i L_{s-i}^{p^{ni}}}, \qquad \begin{bmatrix} s \\ 0 \end{bmatrix} = \frac{F_s}{L_s}, \qquad \begin{bmatrix} s \\ s \end{bmatrix} = 1. \tag{1.4}$$

Thus the polynomial in $u$ occurring in (1.1) closely resembles the polynomial [1]

$$\psi_s(u) = \sum_{i=0}^{s} (-1)^{s-i} \begin{bmatrix} s \\ i \end{bmatrix} u^{p^{ni}}, \tag{1.5}$$

which has the characteristic property

$$\psi_s(u) = \prod_{\deg E < s} (u - E), \tag{1.6}$$

the product extending over all polynomials $E$ (including 0) of degree $< s$. A closer connection will appear below.

If now we put

$$M \equiv A^{p^{n(s-1)}} \pmod P,$$

as may always be done, we shall derive the following criterion [2] for the congruence (1.1): Let


* Received December 11, 1936.


[1] See *Duke Mathematical Journal*, vol. 1 (1936), pp. 139-142; this paper will be cited as DJ. For the congruence $\psi_s(u) \equiv M$, see *Bulletin of the American Mathematical Society*, vol. 41 (1935), pp. 907-914.

[2] For the case $s = 1$, see DJ, pp. 164-168.

$$(1.7)\qquad \begin{aligned} P &= x^k + c_1 x^{k-1} + \cdots + c_k, \qquad (c_j \text{ in } GF(p^n)),\\ P' &= kx^{k-1} + (k-1)c_1 x^{k-2} + \cdots + c_{k-1}, \end{aligned}$$

so that $P'$ is the (formal) derivative of $P$. Assume $k > s$: *then the congruence* (1.1) *is solvable if and only if the product $AP'$ is congruent* (mod $P$) *to a polynomial of degree* $< k - s$. If this condition is satisfied the congruence has precisely $p^{ns}$ solutions.

**2. The polynomials $g_s(u)$ and $f_s(u)$.** We denote by $g_s(u)$ the polynomial in the left member of (1.1). Since, by (1.2),

$$[s] = [s-i] + [i]^{p^{n(s-i)}},$$

it is evident from (1.3) and (1.4) that

$$\begin{bmatrix} s \\ s-i \end{bmatrix} = \frac{[s]F_{s-1}^{p^n}}{F_{s-i}L_i^{p^{n(s-i)}}} = \frac{F_{s-1}^{p^n}}{F_{s-i-1}^{p^n}L_i^{p^{n(s-i)}}} + \frac{F_{s-1}^{p^n}}{F_{s-i}L_{i-1}^{p^{n(s-i)}}},$$

so that

$$(2.1)\qquad \begin{bmatrix} s \\ s-i \end{bmatrix} = \begin{bmatrix} s-1 \\ s-i-1 \end{bmatrix}^{p^n} + F_{s-1}^{p^n-1} \begin{bmatrix} s-1 \\ s-i \end{bmatrix},$$

for $0 < s - i \leqq s$: by properly defining our symbols we may assert that (2.1) holds also for $s - i = 0, s$. Then, by substituting in the left member of (1.1), we have

$$\begin{aligned} g_s(u) &= \sum_{i=0}^{s} (-1)^i \left\{ \begin{bmatrix} s-1 \\ s-i-1 \end{bmatrix}^{p^n} + F_{s-1}^{p^n-1} \begin{bmatrix} s-1 \\ s-i \end{bmatrix} \right\}^{p^{n(i-1)}} u^{p^{ni}} \\ &= \sum_{i=0}^{s-1} (-1)^i \begin{bmatrix} s-1 \\ s-i-1 \end{bmatrix}^{p^{ni}} u^{p^{ni}} \\ &\qquad - \sum_{i=0}^{s-1} (-1)^i \begin{bmatrix} s-1 \\ s-i-1 \end{bmatrix}^{p^{ni}} F_{s-1}^{(p^n-1)p^{ni}} u^{p^{n(i+1)}}, \end{aligned}$$

from which it follows that[8]

$$(2.2)\qquad g_s(u^{p^n}) = g_{s-1}^{p^n}(u) - g_{s-1}^{p^n}(F_{s-1}^{1-p^{-n}} u^{p^n}) = g_{s-1}^{p^n}(u - F_{s-1}^{1-p^{-n}} u^{p^n}).$$

If we define $g_0(u) = u$, it is clear that (2.2) holds for all $s \geqq 1$.

We next define the polynomial $f_s(u)$ by means of

$$(2.3)\qquad f_s(u) = \sum_{i=0}^{k-s} \frac{F_{i+s-1}}{F_{s-1}F_i^{p^{n(s-1)}}} u^{p^{ni}} \qquad (1 \leqq s \leqq k),$$

where $k$ is some fixed integer $> 0$. Then we have

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[8] The quantity $F_{s-1}^{p^{-n}}$ may be defined in terms of the symbol $x^{p^{-n}}$. Otherwise the formula (2.2) may be interpreted as a congruence (mod $P$), in which case no new symbol is required; this interpretation is sufficient for the application.

$$f_s(u) - F_{s-1}^{p^n-1} f_s^{p^n}(u) = \sum_{i=0}^{k-s} \frac{F_{i+s-1}}{F_{s-1}F_i^{p^{n(s-1)}}} u^{p^{ni}} - \sum_{i=0}^{k-s} \frac{F_{i+s-1}^{p^n}}{F_{s-1}F_i^{p^{ns}}} u^{p^{n(i+1)}}$$
$$= \sum \left\{ \frac{F_{i+s-1}}{F_{s-1}F_i^{p^{n(s-1)}}} - \frac{F_{i+s-2}^{p^n}}{F_{s-1}F_{i-1}^{p^{ns}}} \right\} u^{p^{ni}}$$
$$= \sum \frac{F_{i+s-2}^{p^n}}{F_{s-1}F_i^{p^{n(s-1)}}} \{[i+s-1] - [i]^{p^{n(s-1)}}\} u^{p^{ni}}$$
$$= \sum \frac{F_{i+s-2}^{p^n}}{F_{s-2}^{p^n}F_i^{p^{n(s-1)}}} u^{p^{ni}},$$

from which follows the formula

$$(2.4) \qquad f_s(u^{p^n}) - F_{s-1}^{p^n-1} f_s^{p^n}(u^{p^n}) = f_{s-1}^{p^n}(u),$$

for $s \geqq 2$. Now by (2.3),

$$f_1(u^{p^n}) - F_0^{p^n-1} f_1^{p^n}(u^{p^n}) = \sum_{i=0}^{k-1} u^{p^{n(i+1)}} - \sum_{i=0}^{k-1} u^{p^{n(i+2)}}$$
$$= u^{p^n} - u^{p^{n(k+1)}} = (u - u^{p^{nk}})^{p^n}.$$

Thus if we define $f_0(u)$ by means of

$$(2.5) \qquad f_0(u) = u - u^{p^{nk}},$$

it is evident that (2.4) holds for all $s \geqq 1$.

Making use of the formulas (2.2) and (2.4), we now prove the identical congruence

$$(2.6) \qquad f_s(g_s(u)) \equiv u - u^{p^{nk}} \pmod{P},$$

where $P$ is irreducible of degree $k$.

For $s = 0$, (2.6) follows at once from (2.5) and $g_0(u) = u$. For $s = 1$,

$$g_1(u) = u - u^{p^n}, \quad f_1(u) = u + u^{p^n} + \cdots + u^{p^{n(k-1)}}, \quad f_1(g_1(u)) = u - u^{p^{nk}},$$

so that (2.6) holds in this case also. We now assume that (2.6) holds up to and including the value $s - 1$. Clearly [4]

$$\{g_s(u^{p^n})\}^{1/p^n} = g^{p^{-n}}(u^{p^n})$$

is uniquely determined (mod $P$). Then, by (2.2) and (2.4), we have for $s \geqq 1$,

$$(2.7) \qquad f_s(g_s(u^{p^n})) - F_{s-1}^{p^n-1} f_s^{p^n}(g_s(u^{p^n})) = f_{s-1}^{p^n}(g_s^{p^{-n}}(u^{p^n}))$$
$$= f_{s-1}^{p^n}\{g_{s-1}(u - F_{s-1}^{1-p^{-n}} u^{p^n})\}$$

$$(2.8) \qquad \equiv \{(u - F_{s-1}^{1-p^{-n}} u^{p^n}) - (u - F_{s-1}^{1-p^{-n}} u^{p^n})^{p^{nk}}\}^{p^n},$$

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[4] Compare footnote 3.

since (2.6) is assumed to hold for $s-1$. We rewrite (2.8) in the form

$$u^{p^n} - u^{p^{n(k+1)}} - (F_{s-1}^{p^n-1} u^{p^{2n}} - F_{s-1}^{(p^n-1)p^{nk}} u^{p^{n(k+2)}})$$

or

$$u^{p^n} - u^{p^{n(k+1)}} - F_{s-1}^{p^n-1} (u^{p^n} - u^{p^{n(k+1)}})^{p^n}.$$

If now we compare this with the left member of (2.7) and replace $u^{p^n}$ by $u$, we get

$$f_s(g_s(u)) \equiv cF_{s-1} + u - u^{p^{nk}} \pmod P,$$

where $c$ is in $GF(p^n)$; but from the form of $f_s(u)$ and $g_s(u)$ it is clear that $c = 0$, so that (2.5) holds for the value $s$.

In a similar way we may prove the identical congruence

$$g_s(f_s(u)) \equiv u - u^{p^{nk}} \pmod P, \tag{2.9}$$

which obviously holds for $s = 0, 1$. Indeed to prove (2.9) note that

$$\begin{aligned} g_s(f_s(u^{p^n})) &\equiv g_{s-1}^{p^n}\{f_s^{p^{-n}}(u^{p^n}) - F_{s-1}^{1-p^{-n}} f_s(u^{p^n})\} \\ &\equiv g_{s-1}^{p^n}\{f_{s-1}(u)\} \equiv (u - u^{p^{nk}})^{p^n} \pmod P, \end{aligned}$$

which completes the induction. From (2.6) and (2.9) follows

THEOREM 1. *For $P$ irreducible of degree $k$, and $0 \leqq s \leqq k$,*

$$u - u^{p^{nk}} \equiv f_s(g_s(u)) \equiv g_s(f_s(u)) \pmod P.$$

It is of some interest to observe that (2.6) and (2.9) are equivalent, that is, either implies the other (without the use of formulas (2.2) and (2.4)). This is a consequence of the following

LEMMA. *If $f(u) = \Sigma\, \alpha_j u^{p^{nj}}$, $g(u) = \Sigma\, \beta_j u^{p^{nj}}$, and*

$$f(g(u)) \equiv u - u^{p^{nk}} \pmod P,$$

*where $P$ is irreducible of degree $k$, then also*

$$g(f(u)) \equiv u - u^{p^{nk}} \pmod P.$$

Since the proof is very much like the proof of a similar theorem [5] proved elsewhere, it will be omitted here.

**3. Factorization of $f_s(u)$.** From the identity (1.6) it follows readily that

$$\pi_{k-s}(u) = (-1)^{k-s} \frac{L_{k-s}}{F_{k-s}} \psi_{k-s}(u) = u \prod_{\deg E < k-s} (1 - u/E), \tag{3.1}$$

[5] DJ, p. 152.

the product extending over all $E$ (except 0) of degree $< k - s$. On the other hand, by (1. 5) and the first of (3. 1),

$$\pi_{k-s}^{p^{n(s-1)}}(uL_{k-1}) = L_{k-s}^{p^{n(s-1)}} \sum_{j=0}^{k-1} (-1)^j \frac{L_{k-1}^{p^{n(j+s-1)}}}{F_j^{p^{n(s-1)}} L_{k-s-j}^{p^{n(j+s-1)}}} u^{p^{n(j+s-1)}}.$$

Now, from (1. 3), it is easily seen that

$$\left(\frac{L_{k-1}}{L_{k-s-j}}\right)^{p^{n(j+s-1)}} \equiv (-1)^{j+s-1} F_{j+s-1} \pmod P,$$

$$L_{k-s}^{p^{n(s-1)}} \equiv (-1)^{s-1} \frac{L_{k-1}^{p^{n(s-1)}}}{F_{s-1}},$$

so that

$$\pi_{k-s}^{p^{n(s-1)}}(uL_{k-1}) \equiv L_{k-1}^{p^{n(s-1)}} \sum_{j=0}^{k-s} \frac{F_{j+s-1}}{F_{s-1}F_j^{p^{n(s-1)}}} u^{p^{n(j+s-1)}},$$

and therefore,

$$\left\{ \frac{1}{L_{k-1}} \pi_{k-s}(uL_{k-1}) \right\}^{p^{n(s-1)}} \equiv f_s(u^{p^{n(s-1)}}) \pmod P. \tag{3.2}$$

Comparison of (3. 2) and (3. 1) leads to the factorization

$$f_s(u) \equiv u \prod_{\deg E < k-s} \left\{ 1 - \frac{uL_{k-1}^{p^{n(s-1)}}}{E^{p^{n(s-1)}}} \right\} \pmod P. \tag{3.3}$$

But [6] $L_{k-1} \equiv (-1)^{k-1}P'$, where $P'$ is defined by (1. 7); therefore we may put (3. 3) in the form

$$f_s(u) \equiv u \prod_{\deg E < k-s} \left\{ 1 - \frac{uP'^{p^{n(s-1)}}}{E^{p^{n(s-1)}}} \right\} \pmod P. \tag{3.4}$$

THEOREM 2. *The polynomial $f_s(u)$ factors completely* $(\bmod P)$; *the roots are $(E/P')^{p^{n(s-1)}}$, where $E$ ranges over the $p^{n(k-s)}$ polynomials of degree $< k - s$, and $P'$ is the derivative of $P$.*

**4. Criteria for solvability.** If the congruence

$$g_s(u) \equiv M \pmod P \tag{4.1}$$

is assumed solvable, it follows from (2. 6) that

$$f_s(M) \equiv f_s(g_s(u)) \equiv u - u^{p^{nk}} \pmod P, \tag{4.2}$$

for $P$ irreducible of degree $k$. But by Fermat's Theorem, if $A$ is any quantity $(\bmod P)$, $A^{p^{nk}} \equiv A$, so that (4. 2) implies

$$f_s(M) \equiv 0 \pmod P; \tag{4.3}$$

[6] DJ, p. 166.

that is, (4.3) is a *necessary* condition that (4.1) be solvable. To show that this condition is also sufficient, we make use of Theorem 2 and formula (4.2). By Theorem 2 we have the factorization

$$f_s(g_s(u)) \equiv C \prod_{\delta} (g_s(u) - \delta) \pmod{P}, \tag{4.4}$$

where $\delta$ ranges over the roots of $f_s(\delta) \equiv 0$, and $C$ is independent of $u$. If, now, we compare (4.4) with (4.2) and recall that $u^{p^{nk}} - u$ factors completely into linear factors, it is clear that for all $\delta$ (satisfying the congruence $f_s(\delta) \equiv 0$) the congruence $g_s(u) \equiv \delta$ is solvable; further, since $g_s(u) - \delta$ divides $u^{p^{nk}} - u$ it follows that the congruence in question has the maximum number of solutions. We may now state

THEOREM 3. *The congruence* $g_s(u) \equiv M \pmod{P}$, *where* $P$ *is irreducible of degree* $k > s$, *is solvable if and only if* $f_s(M) \equiv 0 \pmod{P}$. *If this condition is satisfied, the congruence has precisely* $p^{ns}$ *solutions.*

Now by Theorem 2, the roots of $f_s(u) \equiv 0$ are the quantities $(E/P')^{p^{n(s-1)}}$, where $E$ is of degree $< k - s$. Thus it is necessary that $M$ be congruent to one of these quantities. If then we replace $M$ by $A^{p^{n(s-1)}}$ (clearly $M$ *uniquely* determines $A$), we have the

THEOREM 4. *If* $P$ *is irreducible of degree* $k > s$, *the congruence*

$$g_s(u) \equiv A^{p^{n(s-1)}} \pmod{P} \tag{4.5}$$

*is solvable if and only if the product* $AP'$ *is congruent* $(\operatorname{mod} P)$ *to a polynomial of degree* $< k - s$. *If* $u_0$ *is a particular solution of* (4.5), *then the general solution is* $u_0 + \rho$, *where* $\rho$ *ranges over the* $p^{ns}$ *roots of* $g_s(u) \equiv 0 \pmod{P}$.

The second part of the theorem follows at once from the observation that $g_s(u) \equiv A^{p^{n(s-1)}}$ and $g_s(v) \equiv A^{p^{n(s-1)}}$ imply $g_s(u - v) \equiv 0$. We shall now determine the roots of $g_s(u) \equiv 0$.

**5. The roots of $g_s(u) \equiv 0$.** For $s = 1$, $g_1(u) = u - u^{p^n}$, and the roots are evidently the $p^n$ elements of the $GF(p^n)$.

For $s = 2$, we make use of the recurrence (2.2). Thus we have the condition

$$g_2(u^{p^n}) \equiv g_1^{p^n}(u - F_1^{1-p^{-n}} u^{p^n}) \equiv 0. \tag{5.1}$$

Therefore, by the preceding paragraph,

$$u - F_1^{1-p^{-n}} u^{p^n} \equiv c \qquad (c \text{ in } GF(p^n)),$$

so that

$$F_1 u^{p^n} - (F_1 u^{p^n})^{p^n} \equiv c(x^{p^n} - x),$$

and, therefore, we have at once $F_1 u^{p^n} \equiv cx + c'$, where $c$ and $c'$ are arbitrary elements of $GF(p^n)$. Thus by (5.1) the roots of $g_2(u) \equiv 0 \pmod{P}$ are furnished by $(cx + c')/F_1$.

For the case $s = 3$, we again employ (2.2)

$$g_3(u^{p^n}) \equiv g_2^{p^n}(u - F_2^{1-p^{-n}} u^{p^n}) \equiv 0; \tag{5.2}$$

then, as above, we get

$$u - F_2^{1-p^{-n}} u^{p^n} \equiv \frac{cx + c'}{F},$$

$$F_2 u^{p^n} - (F_2 u^{p^n})^{p^n} \equiv (cx^{p^n} + c')(x^{p^{2n}} - x),$$

from which follows easily

$$F_2 u^{p^n} \equiv cx^{p^n+1} + c'(x^{p^n} + x) + c'',$$

where $c$, $c'$, $c''$ are in $GF(p^n)$; thus by (5.2) the roots of $g_3(u) \equiv 0 \pmod{P}$ are furnished by

$$\frac{cx^{p^n+1} + c'(x^{p^n} + x) + c''}{F_2},$$

where $c$, $c'$, $c''$ independently range over the elements of $GF(p^n)$.

It is now not difficult to determine the roots of $g_s(u) \equiv 0$. Let

$$\sigma_j = \sigma_j^{(s)} = \sigma_j(x, x^{p^n}, \cdots, x^{p^{n(s-1)}}) \tag{5.3}$$

denote the $j$-th elementary symmetric function of the quantities $x, x^{p^n}, \cdots, x^{p^{n(s-1)}}$; thus we have the identity

$$(t + x)(t + x^{p^n}) \cdots (t + x^{p^{n(s-1)}}) = \sum_{j=0}^{s} \sigma_j^{(s)} t^{s-j}. \tag{5.4}$$

We shall prove

THEOREM 5. *The $p^{ns}$ roots of $g_s(u) \equiv 0 \pmod{P}$ are furnished by*

$$\rho = \frac{c_1 \sigma_{s-1}^{(s-1)} + c_2 \sigma_{s-2}^{(s-1)} + \cdots + c_s \sigma_0^{(s-1)}}{F_{s-1}}, \tag{5.5}$$

*where the $c_j$ independently range over the elements of $GF(p^n)$, and $\sigma_j^{(s-1)}$ is defined by* (5.3).

The theorem is evidently true for $s = 1, 2, 3$. Assuming it to hold up to and including the value $s - 1$, we use (2.2)

$$g_s(u^{p^n}) \equiv g_{s-1}^{p^n}(u - F_{s-1}^{1-p^{-n}} u^{p^n}) \equiv 0.$$

Thus since the theorem is assumed true for the case $s-1$, we have at once

$$F_{s-1}u^{p^n} - (F_{s-1}u^{p^n}) = (c_1\sigma_{s-2}^{(s-2)} + \cdots + c_{s-1}\sigma_0{}^{(s-2)})^{p^n}(x^{p^{n(s-1)}} - x).$$

It is clear that, to complete the induction, it is only necessary to show that

$$\begin{aligned}(5.6) \qquad (c_1\sigma_{s-1}^{(s-1)} + \cdots + c_s\sigma_0{}^{(s-1)})^{p^n} - (c_1\sigma_{(s-1)}^{s-1} + \cdots + c_s\sigma_0{}^{(s-1)}) \\ = (c_1\sigma_{s-2}^{(s-2)} + \cdots + c_{s-1}\sigma_0{}^{(s-2)})(x^{p^{n(s-1)}} - x).\end{aligned}$$

From (5.4), it follows that

$$(t + x^{p^n})(t + x^{p^{2n}}) \cdots (t + x^{p^{ns}}) = \sum_{j=0}^{s} (\sigma_j{}^{(s)})^{p^n} t^{s-j};$$

combining this with (5.4) we have

$$(x^{p^{ns}} - x) \cdot (t + x^{p^n}) \cdots (t + x^{p^{n(s-1)}}) = \sum_{j=0}^{s} \{(\sigma_j{}^{(s)})^{p^n} - \sigma_j{}^{(s)}\} t^{s-j}.$$

But this implies

$$(\sigma_j{}^{(s)})^{p^n} - \sigma_j{}^{(s)} = (\sigma_{j-1}^{(s-1)})^{p^n}(x^{p^{ns}} - x);$$

in this formula replace $s$ by $s-1$ and (5.6) follows immediately. This completes the proof of the theorem.

As an immediate corollary of Theorem 5 and the latter part of Theorem 4, we state

THEOREM 6. *If $u_0$ is a particular solution of $g_s(u) \equiv A^{p^{n(s-1)}}$ mod $P$), then the general solution is $u_0 + \rho$, where $\rho$ is determined by* (5.5).

**6. Some extensions.** If $f_s(A^{p^{n(s-1)}}) \equiv 0 \pmod{P}$, it is clear from (2.4) that also $f_{s-1}(A^{p^{n(s-2)}}) \equiv 0$; however, the converse is not true in general. Thus it may happen that the congruence

$$g_s(u) \equiv A^{p^{n(s-1)}} \pmod{P}$$

is not solvable, while the congruence

$$g_{s-1}(u) \equiv A^{p^{n(s-2)}} \pmod{P}$$

is solvable. In this case it is easily seen that $g_s(u) - A^{p^{n(s-1)}}$ breaks up (mod $P$) into a product of $p^{n(s-1)}$ factors each of degree $p^n$. This follows from the formula

$$(6.1) \qquad g_s(u) - A^{p^{n(s-1)}} = \{g_{s-1}(u^{p^{-n}} - F_{s-1}^{1-p^{-n}} u) - A^{p^{n(s-2)}}\}^{p^n},$$

which is an immediate consequence of (2.2). Assume $f_{s-1}(A^{p^{n(s-2)}}) \equiv 0$, so that by Theorem 3 we have the factorization

$$g_{s-1}(u) - A^{p^{n(s-2)}} \equiv (-1)^{s-1} \frac{F_{s-1}}{L_{s-1}} \prod_{\delta} (u - \delta) \pmod P,$$

where $\delta$ ranges over the $p^{n(s-1)}$ roots of $g_{s-1}(u) \equiv A^{p^{n(s-2)}}$. Substitution in (6.1) gives

$$(6.2) \quad g_s(u) - A^{p^{n(s-1)}} \equiv (-1)^s \left(\frac{F_{s-1}}{L_{s-1}}\right)^{p^n} \prod_{\delta} (F_{s-1}^{p^n-1} u^{p^n} - u + \delta^{p^n}) \pmod P.$$

More generally if we assume only $f_r(A^{p^{n(r-1)}}) \equiv 0 \pmod P$, where $r < s$, then we may show that $g_s(u) - A^{p^{n(s-1)}}$ factors $(\operatorname{mod} P)$ into a product of $p^{nr}$ polynomials, each of degree $p^{n(s-r)}$. Thus formula (6.2) is the special case $r = s - 1$. We now show that in the general case we have a factorization of the form

$$(6.3) \quad g_{r+s}(u) - A^{p^{n(r+s-1)}} \equiv (-1)^{r+s} \left(\frac{F_r}{L_r}\right)^{p^{ns}} \prod_{\delta} \{G_{r,s}(u) - (-1)^s \beta^{p^{ns}}\} \pmod P,$$

where $G_{r,s}(u)$ is a linear [7] polynomial of degree $p^{ns}$, and the product extends over the $p^{nr}$ roots of $g_r(\beta) \equiv A^{p^{n(r-1)}}$.

Let $r$ be fixed; the formula (6.3) is obviously true for $s = 0$. According to (6.2), the formula holds for $s = 1$. Assume that (6.3) holds up to and including the value $s - 1$. Then by (6.1) we have

$$\begin{aligned} g_{r+s}(u) - A^{p^{n(r+s-1)}} &\equiv \{g_{r+s-1}(u^{p^{-n}} - F_{r+s-1}^{1-p^{-n}} u) - A^{p^{n(r+s-2)}}\}^{p^n} \\ &\equiv (-1)^{r+s-1} \left(\frac{F_r}{L_r}\right)^{p^{ns}} \prod_{\beta} \{G_{r,s-1}^{p^n}(u^{p^{-n}} - F_{r+s-1}^{1-p^{-n}} u) - (-1)^{s-1} \beta^{p^{ns}}\} \\ &\equiv (-1)^{r+s} \left(\frac{F_r}{L_r}\right)^{p^{ns}} \prod_{\beta} \{G_{r,s}(u) - (-1)^s \beta^{p^{ns}}\}, \end{aligned}$$

thus completing the induction. It is also evident from the above that the polynomial $G_{r,s}(u)$ satisfies the recurrence

$$(6.4) \qquad G_{r,s}(u) = G_{r,s-1}^{p^n}(F_{r+s-1}^{1-p^{-n}} u - u^{p^{-n}}), \qquad G_{r,0}(u) = u.$$

We have therefore proved

THEOREM 7. *If $f_r(A^{p^{n(r-1)}}) \equiv 0 \pmod P$, the polynomial $g_{r+s}(u) - A^{p^{n(r+s-1)}}$ has the factorization* (6.3), *where $G_{r,s}(u)$ is determined by* (6.4).

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[7] That is, of the form $\Sigma a_j u^{p^{nj}}$.

We shall now show that the polynomial $G_{r,s}(u) - (-1)^s\beta^{p^{ns}}$ occurring in the right member of (6.3) can in general be factored further $(\mathrm{mod}\, P)$; irreducibility occurs only in the case $n = 1 = s$, $f_{r+1}(A^{p^{nr}}) \not\equiv 0 \ (\mathrm{mod}\, P)$.

It is convenient to deal with the left member of (6.3). Let

$$(6.5) \qquad h(u) = g_s(u) - M,$$

where $M$ is arbitrary $(\mathrm{mod}\, P)$. Then by (2.6),

$$(6.6) \qquad f_s\{h(u)\} = f_s\{g_s(u) - M\} \equiv u - u^{p^{nk}} - f_s(M) \ (\mathrm{mod}\, P).$$

In the next place,

$$(6.7) \qquad \begin{aligned} f_s\{h(u)\} - f_s^{p^{nk}}\{h(u)\} &\equiv u - 2u^{p^{nk}} + u^{p^{2nk}} - \{f_s(M) - f_s^{p^{nk}}(M)\} \\ &\equiv u - 2u^{p^{nk}} + u^{p^{2nk}} \ (\mathrm{mod}\, P), \end{aligned}$$

since for arbitrary $M$, $M^{p^{nk}} \equiv M \ (\mathrm{mod}\, P)$. Now put

$$\begin{aligned} u_1 &= u - u^{p^{nk}}, \\ u_2 &= u_1 - u_1^{p^{nk}} = u - 2u^{p^{nk}} + u^{p^{2nk}}, \\ &\cdots\cdots\cdots\cdots \\ u_p &= u_{p-1} - u_{p-1}^{p^{nk}} = u - u^{p^{nkp}}. \end{aligned}$$

Clearly $u_{j+1}$ is a multiple of $u_j$ $(j = 1, \cdots, p-1)$. Thus it follows that $u_2$ divides $u_p$. Therefore by (6.6) and (6.7), the polynomial $h(u)$ is a divisor of $u_p$:

$$(6.8) \qquad h(u) \mid u - u^{p^{nkp}}.$$

Now on the other hand since the set of residues $(\mathrm{mod}\, P)$ form a finite field $GF(p^{nk})$, it follows from a well known theorem that all the irreducible divisors $(\mathrm{mod}\, P)$ of $u_p$ are of degree 1 or $p$. Therefore by (6.8) the same is true of $h(u)$. On the other hand it is evident from Theorem 6 that if $g_s(u) - M$ has one linear factor, then it has $p^{ns}$ linear factors. This proves the following

THEOREM 8. *For arbitrary $M$, the polynomial $g_s(u) - M$* $(\mathrm{mod}\, P)$ *either factors completely into linear factors, or else is a product of $p^{n(s-1)}$ irreducible polynomials each of degree $p$.*

By Theorem 5, if $f_r(M) \not\equiv 0 \ (\mathrm{mod}\, P)$, the polynomial $g_s(u) - M$ has no linear factor. Hence by the preceding theorem we have

THEOREM 9. *If $f_s(M) \not\equiv 0$* $(\mathrm{mod}\, P)$, *then the polynomial $g_s(u) - M$ is a product* $(\mathrm{mod}\, P)$ *of $p^{n(s-1)}$ irreducible factors each of degree $p$.*

Suppose now in (6.5) we put $M \equiv A^{p^{n(r+s-1)}}$. Then comparison with (6.3) leads at once to

THEOREM 10. *Let* $f_r(A^{p^{n(r-1)}}) \equiv 0$, $f_{r+1}(A^{p^{nr}}) \not\equiv 0 \pmod{P}$; *let* $\beta$ *be a root of* $g_r(\beta) \equiv A^{p^{n(r-1)}}$. *Then the polynomial* $G_{r,s}(u) - (-1)^s \beta^{p^{ns}}$ *occurring in the right member of* (6.3) *is a product* (mod $P$) *of* $p^{n(s-1)}$ *irreducible polynomials each of degree* $p$.

In particular for $n = 1$, $s = 1$, we get

THEOREM 11. *If the hypotheses of Theorem* 10 *hold, and in addition* $n = s = 1$, *then the polynomial*

$$G_{r,1}(u) + \beta^p = F_r^{p-1} u^p - u + \beta^p$$

*is irreducible* (mod $P$).

It is not difficult to prove this theorem directly.

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# ON A TRIGONOMETRICAL SERIES OF RIEMANN.*

By Aurel Wintner.

In his paper on Riemann integrals and trigonometrical series, Riemann [1] considers the series

$$\sum_{k=1}^{\infty} \frac{\psi(kx+\frac{1}{2})}{k} \tag{1}$$

and

$$\sum_{n=1}^{\infty} \frac{c(n)}{n} \sin 2\pi nx, \tag{2}$$

where

$$\psi(x) = \begin{cases} x-[x]-\frac{1}{2} & \text{if } x \neq [x] \\ 0 & \text{if } x=[x] \end{cases} \tag{3}$$

and

$$c(n) = \sum_{d|n} (-1)^d, \tag{4}$$

the summation in (4) being extended over the $d(n)$ divisors $d$ of $n$. Riemann's aim in considering the series (1), (2) is to illustrate the limitations to which his definition of an integral subjects the theory of Fourier series. In fact, Riemann observes that if $x$ is rational, both series (1), (2) are convergent and represent the same value, while the function defined by these series on the set of rational numbers is a non-bounded function on the set of those rational numbers which are contained in any fixed interval.

This statement of Riemann has recently been verified by Chowla and Walfisz [2] who discussed the series (1) and (2) for irrational values $x$ as well. They proved, among other things, that the trigonometrical series (2) is almost everywhere convergent and represents almost everywhere the sum of the series (1). That the series (1) is almost everywhere convergent, is an obvious consequence of Khintchine's metrical results concerning diophantine approximations.

The object of the present note is an approach to Riemann's series from the point of view of Lebesgue's theory. It will be seen that the *trigonometrical* series (2) is a *Fourier* series in the sense of Lebesgue and belongs to the

* Received April 28, 1937.

[1] B. Riemann, *Gesammelte mathematische Werke*, 2nd edition, Leipzig, 1892, p. 263.

[2] S. Chowla and A. Walfisz, "Ueber eine Riemannsche Identität," *Acta Arithmetica*, vol. 1 (1935), pp. 87-112.

function defined by (1), i. e., that the odd periodic function (1) is integrable in the sense of Lebesgue and has the coefficients of (2) as Fourier constants. It will also be shown that not only the function (1) but also every positive power of it is integrable in the sense of Lebesgue, i. e., that *the function* (1) *is of class $L^p$ for arbitrarily large $p$*, although this function is non-bounded in every interval (also if one discards sets of measure zero). It has, perhaps, a historical interest that the Lebesgue theory of integration and of Fourier series applies without difficulty to the example by means of which Riemann himself illustrated the limitations of his theory.

That (2) *is almost everywhere convergent* to the function (1), will turn out to be an immediate consequence of the fact [3] that if a Fourier series

$$(5_1) \qquad f(x) \sim \Sigma\,(a_n \cos nx + b_n \sin nx)$$

is such that

$$(5_2) \qquad \Sigma\, n^\delta (a_n{}^2 + b_n{}^2) < +\infty, \text{ where } \delta > 0,$$

then $(5_1)$ is convergent almost everywhere to the function $f(x)$. The exceptional $x$-set of measure zero is, according to Chowla and Walfisz,[2] such that its elements $x$ essentially depend on the arithmetical structure of the number $x$.

First, it will be shown that the series

$$(6) \qquad \sum_{m=1}^{\infty} \frac{\psi(mx)}{m}$$

is convergent in the mean, i. e., that

$$(7) \qquad \lim_{\substack{n\to\infty \\ j\to\infty}} \int_0^1 \{f_{n+j}(x) - f_n(x)\}^2\, dx = 0,$$

where $f_n(x)$ denotes the odd periodic $R$-integrable function

$$(8) \qquad f_n(x) = \sum_{m=1}^{n} \frac{\psi(mx)}{m}.$$

Since the Fourier series of the Bernoulli polynomial (3) is

$$(9) \qquad \psi(x) \sim -\frac{1}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} \sin 2\pi kx,$$

it is seen from (8) that

[3] According to Kolmogoroff and Seliverstoff, one can replace $n^\delta$ in $(5_2)$ by $\log n$; cf. A. Kolmogoroff and G. Seliverstoff, "Sur la convergence des séries de Fourier," *Rendiconti della Reale Accademia Nazionale dei Lincei*, ser. 6, vol. 3 (1926), pp. 307-310.

$$f_n(x) \sim -\frac{1}{\pi}\sum_{k=1}^{\infty}\left(\frac{1}{k}\sum_{d|k}^{1\leqq d\leqq n} 1\right)\sin 2\pi kx, \tag{10}$$

where the inner sum is extended over those divisors $d$ of $k$ which are not greater than $n$. Thus, for every positive integer $j$,

$$f_{n+j}(x) - f_n(x) \sim -\frac{1}{\pi}\sum_{k=1}^{\infty}\left(\frac{1}{k}\sum_{d|k}^{n<d\leqq n+j} 1\right)\sin 2\pi kx.$$

Hence it is seen from Parseval's relation that (7) is equivalent to

$$\lim_{\substack{n\to\infty\\ j\to\infty}}\sum_{k=1}^{\infty}\left(\frac{1}{k}\sum_{d|k}^{n<d\leqq n+j} 1\right)^2 = 0,$$

i. e., to

$$\lim_{n\to\infty}\sum_{k=n+1}^{\infty}\left(\frac{1}{k}\sum_{d|k} 1\right)^2 = 0.$$

Since $\sum_{d|k} 1$ is the number $d(k)$ of divisors of $k$, it follows that one merely has to show the convergence of the series

$$\sum_{k=1}^{\infty}\frac{d(k)^2}{k^2}.$$

Now

$$\sum_{k=1}^{\infty}\frac{d(k)^2}{k^\beta} \tag{11}$$

is convergent for every $\beta > 1$, since $|\zeta(\sigma+it)|^4$ has the finite mean value

$$\lim_{T\to\infty}\frac{1}{T}\int_1^T |\zeta(\sigma+it)|^4\,dt = \sum_{k=1}^{\infty}\frac{d(k)^2}{k^{2\sigma}}$$

for[4] every $\sigma > \frac{1}{2}$.

By a well-known theorem of Fischer, the relation (7) just proved implies the existence of a function $f(x)$ which belongs to the class $L^2$ and is such that

$$\lim_{n\to\infty}\int_0^1 \{f(x) - f_n(x)\}^2\,dx = 0. \tag{12}$$

Since the functions (8) are the partial sums of the series (6), the relation (12)

[4] Cf., e. g., E. C. Titchmarsh, *The zeta-function of Riemann*, Cambridge, 1930, pp. 38-41.

may be expressed by saying that the series (6) converges in the mean to a function $f(x)$ of class $L^2$.

It is well known that (12) implies the existence of a subsequence of $\{f_n(x)\}$ such that this subsequence of $\{f_n(x)\}$ tends almost everywhere to $f(x)$. In other words, there exists an increasing sequence $\{\mu_h\}$ of positive integers such that if one unites the first $\mu_1$, then the next $\mu_2$ terms of the series (6), and so on, the resulting "bracketed" series converges almost everywhere to $f(x)$. Actually, the introduction of the brackets is superfluous in view of Khintchine's result referred to above. This fact will not be needed in what follows. For (12) in itself allows one to consider (6) as the definition of an odd periodic function $f(x)$ of class $L^2$, this function being undetermined on a set of measure zero.

Since the functions (8) tend in the mean to the function (6) of class $L^2$, the $k$-th Fourier constant of (8) tends, as $n \rightarrow \infty$, to the $k$-th Fourier constant of (6) for every fixed $k$. Hence it is seen from (10) that the $k$-th Fourier constant of (6) is

$$\lim_{n\rightarrow\infty} -\frac{1}{\pi k} \sum_{d|k}^{1 \leqq d \leqq n} 1 = -\frac{1}{\pi k} \sum_{d|k} 1 = \frac{d(k)}{-\pi k}.$$

In other words,

$$\sum_{m=1}^{\infty} \frac{\psi(mx)}{m} \sim -\frac{1}{\pi} \sum_{k=1}^{\infty} \frac{d(k)}{k} \sin 2\pi kx. \tag{13}$$

It follows that the function (6) is of class $L^p$ not only for $p = 2$ but for arbitrarily large $p$. In fact, on applying to (13) Hausdorff's extension of the Fischer-Riesz theorem, it is seen that (6) is of class $L^p$ for every $p$ if

$$\sum_{k=1}^{\infty} \left( \frac{d(k)}{k} \right)^{p/(p-1)}$$

is convergent for every $p \geqq 2$. Thus it is sufficient to show that

$$\sum_{k=1}^{\infty} \left( \frac{d(k)}{k} \right)^{1+\epsilon}$$

is convergent for every positive $\epsilon \leqq 1$. Since $d(k) \geqq 1$, it follows that it is sufficient to know the convergence of

$$\sum_{k=1}^{\infty} \frac{d(k)^2}{k^{1+\epsilon}}$$

for every $\epsilon > 0$. Since (11) is convergent for every $\beta > 1$, the proof is complete.

The convergence of (11) for $\beta > 1$ also implies that

$$\sum_{k=1}^{\infty} k^{\delta} \left(\frac{d(k)}{k}\right)^2$$

is convergent for sufficiently small values of $\delta > 0$. Hence, on comparing (13) with $(5_2)$, it is seen from the criterion $(5_1)$ that the Fourier series (13) is convergent almost everywhere. Since the arithmetical means of a Fourier series tend almost everywhere to the function to which it belongs, the sum of the Fourier series (13) is almost everywhere equal to the function (6).

The above results concern not Riemann's series (1), (2) but the function (6). The transition to Riemann's series can be based on the Bernoulli identity

$$\psi(x + \tfrac{1}{2}) = \psi(2x) - \psi(x), \tag{14}$$

which is obvious from the definition (3).

First, since (6) is an odd periodic function of class $L^p$, where $p$ is arbitrarily large, the same holds for the function

$$\sum_{m=1}^{\infty} \frac{\psi(m2x)}{m}.$$

It follows, therefore, from Hölder's inequality that the function

$$\sum_{m=1}^{\infty} \frac{\psi(2mx)}{m} - \sum_{m=1}^{\infty} \frac{\psi(mx)}{m} \tag{15}$$

is of class $L^p$ for every $p$. Now (15) is, in view of (14), identical with Riemann's function (1). Furthermore, it is seen from (13) that the Fourier series of the difference (15) is

$$\frac{-1}{\pi} \sum_{k=1}^{\infty} \frac{d(k)}{k} \sin 4\pi kx + \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{d(k)}{k} \sin 2\pi kx. \tag{16}$$

Now $d(k) = \sum_{d|k} 1$, so that the Fourier series (16) is, in view of (4), identical with Riemann's trigonometrical series (2). Accordingly, Riemann's function (1) is of class $L^p$ for every $p$ and has the Fourier series

$$\sum_{m=1}^{\infty} \frac{\psi(mx + \frac{1}{2})}{m} \sim -\frac{1}{\pi} \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{d|n} (-1)^d\right) \sin 2\pi nx. \tag{17}$$

Finally, the Fourier series (17) converges almost everywhere to the function

(1). In fact, (1) is identical with (15), while the Fourier series (13) converges almost everywhere to the function (6).

While $(\Psi(x))^n$, where $\Psi(x)$ denotes Riemann's function (1), is for every $n$ integrable in the sense of Lebesgue, the function $\Psi(x)$ lies in every sense outside of the range of Riemann's integration theory. In fact, if $\iota$ is any subinterval of the interval $0 \leqq x \leqq 1$, there cannot exist a constant $M = M_\iota$ such that $|\Psi(x)| \leqq M$ almost everywhere in $\iota$. For suppose, if possible, that there exists an $M = M_\iota$ for some $\iota$. Then, if $\lambda$ is any closed interval contained in the interior of $\iota$, one can find a constant $K = K_\lambda$ such that $|S_n(x)| \leqq K$ for every $x$ in $\lambda$ and for every $n$, where the $S_n(x)$ denote the arithmetical means of the Fourier series (17) of $\Psi(x)$. In particular, the $S_n(x)$ are uniformly bounded on the set of rational $x$ contained in $\lambda$. This clearly contradicts the fact mentioned by Riemann [1] and verified by Chowla and Walfisz.[2] It follows, in particular, that if a function $F(x)$ is almost everywhere equal to Riemann's function (1), then $F(x)$ is discontinuous at every $x$.

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# ON DIVERGENT INFINITE CONVOLUTIONS.*

By E. R. van Kampen and Aurel Wintner.

**Introduction.** It is known that the distribution theory of sums of independent random variables can be developed from several points of view which are, however, in the main, equivalent. One, and possibly the most general, approach is represented by Kolmogoroff's axiomatic treatment [1] of questions of probability distribution. This approach applies, in particular, to the problem of probable convergence of series of independent random variables, as first solved by Khintchine and Kolmogoroff.[2] It also implies the treatment based on the Lebesgue measure theory of infinite product spaces, as developed by Steinhaus, Littlewood, Paley and Zygmund, Jessen, and others.[3]

It is known [4] that the main result of Khintchine and Kolmogoroff, which is based on the notion of "equivalent series," can be formulated also in terms of infinite convolutions. The results of the present paper concern infinite convolutions and imply, among other things, certain facts which are equivalent to theorems concerning the divergence problem of series of independent random variables. In particular, the results imply essential refinements of certain facts indicated by Lévy.[5] Since Lévy's statements will not be used, the following considerations imply detailed proofs for them.

Theorem 1, which applies not only to convolution sequences, seems to have an independent interest. Theorems 3 and 5 delimit all possibilities which can

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[1] A. Kolmogoroff, "Grundbegriffe der Wahrscheinlichkeitsrechnung," *Ergebnisse der Mathematik und ihrer Grenzgebiete*, vol. 2 (Berlin, 1933), no. 3.

[2] A. Khintchine and A. Kolmogoroff, "Ueber Konvergenz von Reihen, deren Glieder durch den Zufall bestimmt werden," *Recueil de la Societé Mathématique de Moscou*, vol. 32 (1925), pp. 668-677; A. Kolmogoroff, "Ueber die Summen durch den Zufall bestimmter zufälliger Grössen," *Mathematische Annalen*, vol. 99 (1928), pp. 309-319; vol. 102 (1930), pp. 484-488.

[3] Cf. P. Lévy, "Sur quelques points de la théorie des probabilités dénombrables," *Annales de l'Institut Henri Poincaré*, vol. 6 (1936), pp. 153-184, where further references are given.

[4] B. Jessen and A. Wintner, "Distribution functions and the Riemann zeta function," *Transactions of the American Mathematical Society*, vol. 38 (1935), pp. 48-88, more particularly 84-86.

[5] P. Lévy, "Sur les séries dont les termes sont des variables eventuelles indépendentes," *Studia Mathematica*, vol. 3 (1931), pp. 119-155, more particularly chap. I; cf. also the corrections on p. 337 of vol. 3 (1934) of the *Annali della R. Scuola Normale Superiore di Pisa*.

actually occur in case of a divergent infinite convolution. Theorem 9 describes what can happen to a divergent or not absolutely convergent infinite convolution upon a reordering of its "factors." Due to the correspondence alluded to above,[4] a part of Theorem 3 can be interpreted as a manifestation of the famous "0 or 1" principle.[6] The total content of Theorem 3, which concerns infinite convolutions, cannot conveniently be formulated in terms of the probable convergence or divergence of sums of independent random variables, or in terms of a Lebesgue measure of an infinite product space.

All distribution problems under consideration will be assumed to be one-dimensional, so that the random variables are real numbers.

**Metric.** In what follows, Greek letters $\phi, \rho, \cdots$ will denote monotone non-decreasing functions of a real variable $x$ which remain bounded as $x \to \pm\infty$. The case of a constant function $\alpha$, i. e., the case where $\phi(x) = \alpha$ for every $x$ and for some real number $\alpha$, is not excluded.

(I) For a given function $y = \phi(x)$, the symbol $]\phi[$ will denote the bounded open interval $\phi(-\infty) < y < \phi(+\infty)$ or the point $y = \phi(-\infty)$ according as $\phi(-\infty) < \phi(+\infty)$ or $\phi(-\infty) = \phi(+\infty)$, i. e., according as $\phi(x)$ is not or is a constant function.

It will be convenient to consider a function $y = \phi(x)$ as a Jordan curve in an $(x, y)$-plane. This is made possible by adjoining the point $(x, y) = (x, \phi(x))$ to the segment constituted by the set of points $(x, y)$, where $y$ describes the closed interval $\phi(x - 0) \leqq y \leqq \phi(x + 0)$, if $x$ is a discontinuity point of $\phi$. Thus two functions, $\phi$ and $\psi$, determine the same Jordan curve if and only if the two functions are equal at their continuity points, i. e., if $\phi(x + 0) = \psi(x + 0)$ and/or $\phi(x - 0) = \psi(x - 0)$ for every $x$. In this case the functions $\phi$ and $\psi$ will be considered as identical. Correspondingly, a sequence of functions $\phi_n(x)$ is said to be convergent if there exists a function $\phi(x)$ such that $\phi_n(x) \to \phi(x)$ holds at every continuity point $x$ of $\phi(x)$. The signs $=$ and $\to$ will only be used in the sense just defined. By a classical theorem of Helly, $\phi_n \to \phi$ whenever $\phi_n(x) \to \phi(x)$ holds for a dense set of values $x$.

For a given number $\epsilon > 0$, let the $\epsilon$-strip about the Jordan curve $y = \phi(x)$ be defined as the set of those points of the $(x, y)$-plane whose distance from at least one point of the Jordan curve is less than $\epsilon$.

(II) For two functions $y = \phi_1(x)$, $y = \phi_2(x)$, let $|\phi_1; \phi_2|$ or $|\phi_1(x); \phi_2(x)|$ denote the greatest lower bound of those $\epsilon > 0$ for which every

[6] A. Kolmogoroff, *loc. cit.* [1], pp. 60-61.

point of the Jordan curve $y = \phi_1(x)$ is contained in the $\epsilon$-strip about the Jordan curve $y = \phi_2(x)$.

Thus $|\phi_1 ; \phi_2|$ is a non-negative number not less than

$$\operatorname{Max}\left(|\phi_1(+\infty) - \phi_2(+\infty)|,\ |\phi_1(-\infty) - \phi_2(-\infty)|\right)$$

and not greater than

$$\operatorname{Max}\left(\phi_1(+\infty) - \phi_1(-\infty),\ \phi_2(+\infty) - \phi_2(-\infty)\right).$$

It is easily verified that

$$(1_1) \qquad |\phi_1 ; \phi_2| = 0 \text{ if and only if } \phi_1 = \phi_2;$$

$$(1_2) \qquad |\phi_1 ; \phi_2| = |\phi_2 ; \phi_1|;$$

$$(1_3) \qquad |\phi_1 ; \phi_2| \leqq |\phi_1 ; \phi_3| + |\phi_3 ; \phi_2|.$$

This means that (II) defines a metrization of the space of all functions $\phi(x)$, if the sign $\phi_1 = \phi_2$ is defined as above.[7] This metrization is easily seen to be a complete metrization, in the sense that

$$(2) \qquad \lim_{\substack{n=\infty \\ m=\infty}} |\phi_n ; \phi_m| = 0 \text{ if and only if } \lim_{n=\infty} |\phi_n ; \phi| = 0 \text{ for a } \phi.$$

On the other hand, it is not true that convergence with reference to the metrization defined by (II) is equivalent to convergence represented above by the symbol $\phi_n \to \phi$. This is shown by the example $\phi_n(x) = \operatorname{sgn}(x + n)$, where $\phi_n \to \phi$ does, but $|\phi_n ; \phi| \to 0$ does not, hold for $\phi(x) \equiv 1$. The general situation is easily seen to be this:

(III) The three conditions

$$(3_1)\ \phi_n \to \phi; \qquad (3_2)\ \phi_n(+\infty) \to \phi(+\infty); \qquad (3_3)\ \phi_n(-\infty) \to \phi(-\infty)$$

are necessary and sufficient for $|\phi_n ; \phi| \to 0$.

**The function set $\|\phi_n\|$.** Two functions $\phi_1(x)$, $\phi_2(x)$ will be said to be congruent if there exists a number $c$ such that the two functions $\phi_1(x)$, $\phi_2(x - c)$ are identical.

(IV) For a given sequence $\{\phi_n\}$ of functions $\phi_n(x)$, let $\|\phi_n\|$ denote the set of those functions $\rho(x)$ for which one can choose a sequence $\{c_n\}$ of numbers $c_n$ such that

$$(4) \qquad \phi_n(x - c_n) \to \rho(x), \qquad n \to \infty,$$

holds at every continuity point $x$ of $\rho$.

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[7] For another metrization, cf. P. Lévy, *loc. cit.* [5], pp. 339-341.

It is clear from this definition that

(5) $\| \phi_n \| \subset \| \psi_m \|$ whenever $\{\psi_m\}$ is a subsequence of $\{\phi_n\}$,

and that

(6) if $\rho(x) \subset \| \phi_n \|$, then $\rho(x-c) \subset \| \phi_n \|$ for every $c$.

(V) If $\phi_n{}^1$ and $\phi_n{}^2$ are congruent for $n = 1, 2, \cdots$, then the two function sets $\| \phi_n{}^1 \|$, $\| \phi_n{}^2 \|$ are identical.

This is clear from the definitions.

Lemma 1. *If $\rho^1$ and $\rho^2$ are contained in a function set $\| \phi_n \|$, then either the two functions $\rho^1$, $\rho^2$ are congruent or the two point sets $]\rho^1[$, $]\rho^2[$, defined under* (I), *have no point in common.*

*Proof.* Suppose, if possible, that $\rho^1$, $\rho^2$ are not congruent and $]\rho^1[$, $]\rho^2[$ do have a point in common. Then, on interchanging, if necessary, $\rho^1$ and $\rho^2$, there clearly exists an $x = x_0$ for which

$$\rho^1(-\infty) < \rho^2(x_0) < \rho^1(+\infty). \tag{7}$$

Since $\rho^1 \subset \| \phi_n \|$ and $\rho^2 \subset \| \phi_n \|$, there exist two sequences of numbers, say $\{c_n{}^1\}$ and $\{c_n{}^2\}$, such that

$$\phi_n(x - c_n{}^1) \to \rho^1(x), \qquad \phi_n(x - c_n{}^2) \to \rho^2(x). \tag{8}$$

Now the sequence of the differences $c_n{}^1 - c_n{}^2$ contains a subsequence which tends either to a finite limit $c$ or to $-\infty$ or to $+\infty$. In the first case (8) implies that $\rho^1(x) = \rho^2(x-c)$, which is a contradiction, since $\rho^1$ and $\rho^2$ are not congruent, by hypothesis. In the second case (8) clearly implies that $\rho^2(x) \leqq \rho^1(-\infty)$. Since this contradicts (7), and since the third case can be treated in the same way as the second case, the proof of Lemma 1 is complete.

Lemma 2. *If $\{\rho_m\}$ is a sequence of functions contained in a function set $\| \phi_n \|$, then $\| \rho_m \| \subset \| \phi_n \|$.*

*Proof.* If $\sigma \subset \| \rho_m \|$ and $\sigma$ is not a constant function, then Lemma 1 implies that $\sigma(x)$ and $\rho_m(x)$ are congruent for every sufficiently large $m$, so that $\sigma \subset \| \phi_n \|$, by (6). Hence it is sufficient to prove that every constant function $\alpha$ contained in $\| \rho_m \|$ is contained in $\| \phi_n \|$. In view of (V), one can assume without loss of generality that $\rho_m(x) \to \alpha$ as $m \to \infty$. Then there exists for every $\epsilon > 0$ and for every $t > 0$ an $M = M(\epsilon, t)$ such that

$$| \rho_m(\pm t) - \alpha | < \epsilon \text{ for every } m \geqq M(\epsilon, t). \tag{9}$$

Furthermore, since $\rho_m \subset \| \phi_n \|$ for $m = 1, 2, \cdots$, there exist constants $c_n{}^m$ $(m, n = 1, 2, \cdots)$ such that

$$\phi_n(x - c_n{}^m) \rightarrow \rho_m(x) \text{ as } n \rightarrow \infty \qquad (m = 1, 2, \cdots).$$

Hence if $t > 0$ is such that neither $x = t$ or $x = -t$ is contained in the at most enumerable set which consists of the points $x$ at which at least one $\rho_m$ is discontinuous, then, by the definition of the symbol $\rightarrow$, one can choose an $N = N(\epsilon, m, t)$ such that

$$| \phi_n(\pm t - c_n{}^m) - \rho_m(\pm t) | < \epsilon \text{ for every } n \geqq N(\epsilon, m, t); \qquad (m = 1, 2, \cdots).$$

Consequently, from (9),

$$(10) \qquad | \phi_n(\pm t - c_n{}^m) - \alpha | < 2\epsilon, \text{ if } m \geqq M(\epsilon, t), \ n \geqq N(\epsilon, m, t).$$

Since $\phi_n(x)$ is monotone and $\alpha$ is independent of $x$, it is clear that (10) remains valid if one replaces $\phi_n(\pm t - c_n{}^m)$ by $\phi_n(x - c_n{}^m)$, where $x$ is any number between $-t$ and $t$. Hence

$$(11) \qquad | \phi_n(x - c_n{}^m) - \alpha | < 2t^{-1}, \text{ if } | x | \leqq t, \ m \geqq M(t), \ n \geqq N(t),$$

where

$$M(t) = M(t^{-1}, t), \ N(t) = N(t^{-1}, M(t), t); \qquad (\epsilon = t^{-1}).$$

One can clearly assume that

$$N(t') < N(t''), \text{ if } t' < t''.$$

Since $t$ is any positive number not belonging to an at most enumerable set, one can choose $t = t_1, t_2, \cdots$, where $t_k \rightarrow \infty$ as $k \rightarrow \infty$. For a fixed $n$ which is not less $N(t_1)$, let the integer $k = k_n$ be defined by the condition

$$N(t_{k_n}) \leqq n < N(t_{k_{n+1}}).$$

Then

$$(12) \qquad k_n \rightarrow \infty \text{ and } t_{k_n} \rightarrow \infty \text{ as } n \rightarrow \infty.$$

Now, on placing

$$d_n = c_n{}^m, \text{ where } m = M(t_{k_n}),$$

it is clear from (11) that

$$| \phi_n(x - d_n) - \alpha | < 2t_{k_n}{}^{-1}, \text{ if } | x | \leqq t_{k_n}, \ n \geqq N(t_{k_n}).$$

Hence (12) implies that

$$\phi_n(x - d_n) \rightarrow \alpha, \qquad n \rightarrow \infty,$$

for every $x$. Thus $\alpha \subset \| \phi_n \|$. This completes the proof of Lemma 2.

(VI) If $\rho(x)$ is contained in $\| \phi_n \|$, then so are the constant functions $\alpha = \rho(-\infty)$ and $\alpha = \rho(+\infty)$.

This is clear from Lemma 2, since if $\rho_m(x) = \rho(x \pm m)$, then $\rho_m \subset \| \phi_n \|$, by (6).

**Distribution functions.** A monotone non-decreasing function $\phi(x)$ is said to be a distribution function if the set $]\phi[$, defined in (I), is the interval $0 < y < 1$, i.e., if $\phi(-\infty) = 0$ and $\phi(+\infty) = 1$. By the spectrum of a distribution function $\phi(x)$ is meant the set of those points $x = x_0$ for which $\phi(x_0 - \epsilon) \neq \phi(x_0 + \epsilon)$ whenever $\epsilon \neq 0$.

The metric defined under (II), when applied to the space of all distribution functions, defines a topology which is equivalent to the one defined by the symbol $\phi_n \to \phi$. In fact, since $(3_2)$ and $(3_3)$ are satisfied if $\phi_n$ and $\phi$ are distribution functions, (III) clearly implies

(VII) If $\phi_1, \phi_2, \cdots$ and $\phi$ are distribution functions, then $\phi_n \to \phi$ if and only if $| \phi_n ; \phi | \to 0$.

An obvious corollary of (VII) is the well-known fact that if $\phi_n$ and $\phi$ are distribution functions and $\phi$ is continuous for $-\infty < x < +\infty$, then $\phi_n$ cannot tend to $\phi$ unless the convergence is uniform for $-\infty < x < +\infty$. It is clear from (2) that (VII) implies

(VIII) If $\phi_1, \phi_2, \cdots$ are distribution functions, then there exists a distribution function $\phi$ satisfying $\phi_n \to \phi$ if and only if $\lim\limits_{\substack{n=\infty \\ m=\infty}} | \phi_n ; \phi_m | = 0$.

Throughout the paper, use will be made of the following notation:

(IX) If $\chi(x)$ is a monotone function for which

$$0 \leqq \chi(-\infty) \leqq \chi(+\infty) \leqq 1,$$

let there be defined for every positive number $t$ a distribution function $[\chi]_t$ by placing

$$[\chi(x)]_t = 0, \text{ if } -\infty < x < -t;$$
$$[\chi(x)]_t = \chi(x), \text{ if } -t \leqq x \leqq t;$$
$$[\chi(x)]_t = 1, \text{ if } t < x < +\infty.$$

Thus $[\chi(x)]_t$ is a distribution function which is defined for every distribution function $\chi(x)$ and for certain functions $\chi(x)$ which are not distribution functions.

(X) If $\{\psi_m\}$ is a sequence of distribution functions and $\rho$ a monotone non-decreasing function which need not be a distribution function, then $\psi_m \to \rho$ if and only if $|[\psi_m]_t; [\rho]_t| \to 0$ for every fixed $t > 0$.

This is clear from (VII) and (IX).

**The set $\| \phi_n \|$ in case of distribution functions $\phi_n$.** If $\phi, \phi_2, \cdots$ are distribution functions, it is easily seen from (IV) that $\| \phi_n \|$ contains the constant functions $\alpha = 0$ and $\alpha = 1$. It is shown by the example

$$\phi_{2n}(x) = \tfrac{1}{2}(1 + \operatorname{sgn} x), \qquad \phi_{2n+1}(x) = \tfrac{1}{2}(1 + \tfrac{1}{\pi} \arctan x)$$

that the set $\| \phi_n \|$ belonging to a sequence $\{\phi_n\}$ of distribution functions need not contain any function distinct from the constant functions $\alpha = 0$ and $\alpha = 1$. In particular, the set $\| \phi_n \|$ belonging to a sequence $\{\phi_n\}$ of distribution functions need not contain a distribution function.

THEOREM 1. *Every sequence $\{\phi_n\}$ of distribution functions contains a subsequence $\{\psi_m\}$ which has the following properties:*

(i) *There exists an at most enumerable set of mutually disjoint open subintervals $\alpha_k < y < \beta_k$ of $0 \leqq y \leqq 1$ such that a constant function is contained in $\| \psi_m \|$ if and only if its value $y$ is not contained in any of these intervals $\alpha_k < y < \beta_k$.*

(ii) *There exists for each of these intervals $\alpha_k < y < \beta_k$ a monotone non-decreasing function $\rho_k(x)$ such that the set $]\rho_k[$, defined in* (I), *is the interval $\alpha_k < y < \beta_k$ and a non-constant function is contained in $\| \psi_m \|$ if and only if it is congruent with $\rho_k(x)$.*

The example mentioned before shows that Theorem 1 becomes false if one replaces a suitably chosen subsequence $\{\psi_m\}$ of $\{\phi_n\}$ by $\{\phi_n\}$ itself. The proof of Theorem 1 will be based on the following facts (XI), (XII), (XIII):

(XI) If $\{\phi_n\}$ is a sequence of distribution functions and $s$ a given number such that $0 \leqq s \leqq 1$, then $\{\phi_n\}$ contains a subsequence $\{\psi_m\}$ such that for some element $\rho^s = \rho^s(x)$ of the function set $\| \psi_m \|$ the number $s$ is contained in $]\rho^s[$.

*Proof of* (XI). The assumptions of (XI) clearly imply that

$$\phi_n(b_n - 0) \leqq s \leqq \phi_n(b_n + 0) \tag{13}$$

for every $n$ and for some number $b = b_n$. Since $\{\phi_n(x + b_n)\}$ is a sequence of distribution functions, it contains a convergent subsequence. Let $\{\psi_m(x)\}$

and $\{a_m\}$ be the corresponding subsequences of $\{\phi_n(x)\}$ and $\{-b_n\}$ respectively and let $\sigma(x)$ denote the limit function of $\{\psi_m(x-a_m)\}$. Since $\sigma(x)$ is contained in $\| \psi_m \|$, so are, by (VI), the constant functions $\sigma(\pm\infty)$. Hence, on placing $\rho^s(x) = \sigma(x)$ or $\rho^s(x) = \sigma(\pm\infty)$ according as the number $s$ is contained in $]\sigma[$ or is equal to $\sigma(\pm\infty)$, the statement of (XI) follows.

(XII) Every sequence $\{\phi_n\}$ of distribution functions contains a subsequence $\{\psi_m\}$ such that there exists for every rational number $r$, where $0 \leqq r \leqq 1$, an element $\rho^r$ of $\| \psi_m \|$ for which $r$ is contained in $]\rho^r[$. The

*Proof of* (XII) follows from (XI) by a straight-forward application of the diagonal principle.

(XIII) If a sequence $\{\psi_m\}$ of distribution functions is such that $\| \psi_m \|$ contains for every rational number $r$, where $0 \leqq r \leqq 1$, an element $\rho^r$ for which $]\rho^r[$ contains $r$, then $\| \psi_m \|$ contains for every real number $y$, where $0 \leqq y \leqq 1$, an element $\rho^y$ for which $]\rho^y[$ contains $y$.

*Proof of* (XIII). In the proof that $\rho^y$ exists for a given $y$, it may be assumed without loss of generality that $y$ is neither contained in a $]\rho^r[$ nor is $y$ a boundary point of a $]\rho^r[$; cf. (VI). It is clear that, under these assumptions,

$$\rho^{r_n}(x) \to y, \qquad n \to \infty, \qquad (-\infty < x < +\infty)$$

whenever the sequence $\{r_n\}$ of rational numbers is such that $r_n \to y$. It follows, therefore, from Lemma 2 that the constant function $\alpha = y$ is contained in $\| \phi_n \|$. Thus the requirement of (XIII) is satisfied by the constant function $\rho^y(x) \equiv y$.

*Proof of Theorem* 1. Let $\{\psi_m\}$ be a subsequence of $\{\phi_n\}$ such that $\| \psi_m \|$ has the property stated under (XII). Then there exists, by (XIII), for every $y$, where $0 \leqq y \leqq 1$, a $\rho^y \subset \| \psi_m \|$ such that $y$ is contained in $]\rho^y[$. If $u$ and $v$ are two distinct $y$-values and neither $\rho^u$ nor $\rho^v$ is a constant function, then Lemma 1 implies that the two open intervals $]\rho^u[$, $]\rho^v[$ are either disjoint or coincident, and that in the latter case $\rho^u$ and $\rho^v$ are congruent. Consequently, there exists in the interval $0 \leqq y \leqq 1$ an at most enumerable set of mutually disjoint open intervals $\alpha_k < y < \beta_k$ such that an open $y$-interval is an interval $\alpha_k < y < \beta_k$ if and only if it is an interval $]\rho^y[$ belonging to some non-constant function $\rho^y$, where $0 \leqq y \leqq 1$. Hence it is clear from (XIII) that if a number $y$, where $0 \leqq y \leqq 1$, is in none of the intervals $\alpha_k < y < \beta_k$, then the

constant function $\alpha = y$ is contained in $\| \psi_m \|$. On combining this with Lemma 2, Theorem 1 follows.

It is understood that the open set formed by the intervals $\alpha_k < y < \beta_k$ of Theorem 1 can be the empty set.

(XIV) If a subsequence $\{\psi_m\}$ of a sequence $\{\phi_n\}$ of distribution functions satisfies the requirements (i), (ii) of Theorem 1, then $\| \psi_m \|$ either contains a distribution function or it contains a constant function $\alpha$, where $0 < \alpha < 1$.

This is clear from Theorem 1.

(XV) If a sequence $\{\phi_n\}$ of distribution functions is such that not every constant function $\alpha$ $(0 \leqq \alpha \leqq 1)$ is contained in $\| \phi_n \|$, then some subsequence $\{\psi_m\}$ of $\{\phi_n\}$ is such that $\| \psi_m \|$ contains a non-constant function.

*Proof.* Suppose that there exists a constant function $\alpha$ $(0 \leqq \alpha \leqq 1)$ which is not contained in $\| \phi_n \|$. Then $\alpha \neq 0$ and $\alpha \neq 1$. Hence there exists a sequence $\{c_n\}$ of numbers such that

$$\phi_n(c_n - 0) \leqq \alpha \leqq \phi_n(c_n + 0). \tag{14}$$

Since the sequence of the distribution functions $\phi_n(x + c_n)$ cannot tend to the constant function $\alpha$, it contains a subsequence which tends to a limit function $\rho(x) \not\equiv \alpha$. Hence it is clear from (14) that $\rho(x)$ is not a constant function. Finally, if $\{\psi_m(x)\}$ is that subsequence of $\{\phi_n(x)\}$ which corresponds to the subsequence of $\{\phi_n(x + c_n)\}$ defining $\rho(x)$, then $\rho \subset \| \psi_m \|$. This completes the proof of (XV).

THEOREM 2. *If a subsequence $\{\psi_k\}$ of a sequence $\{\phi_n\}$ of distribution functions satisfies the requirements of Theorem* 1, *then one can choose for every $\epsilon > 0$ and for every $t > 0$ an $M = M(\epsilon, t) > 0$ such that*

(i) *there exists for every $m > M(\epsilon, t)$ and for every function $\rho \subset \| \psi_k \|$ a number $c = c_m$ for which*

$$|[\psi_m(x - c)]_t ;\ [\rho(x)]_t| < \epsilon; \tag{15}$$

(ii) *there exists for every $m > M(\epsilon, t)$ and every number $c$ a $\rho$ such that* (15) *is satisfied for this $\rho = \rho_m{}^c$.* It is understood that the symbols $|\,;\,|$ and $[\ ]_t$ are those defined under (II) and (IX).

*Proof.* For a fixed $\epsilon > 0$, choose $K = K_\epsilon$ values $y_j$ such that

$$0 = y_1 < \cdots < y_j < \cdots < y_K = 1 \text{ and } y_j - y_{j-1} < \tfrac{1}{2}\epsilon. \tag{16}$$

By Theorem 1, there exists for every $j$ an element $\chi = \chi_j$ of $\| \psi_k \|$ such that the $y$-set $]\chi_j[$ contains the point $y = y_j$. Choose a number $a_j$ such that

$$\chi_j(a_j - 0) \leqq y_j \leqq \chi_j(a_j + 0)$$

and put

$$\tau_j(x) = \chi_j(x + a_j).$$

Then

$$\tau_j(-0) \leqq y_j \leqq \tau_j(+0), \tag{17}$$

and $\tau_j(x)$ is, by (6), contained in $\| \psi_k \|$. It follows, therefore, from (X) that one can choose for every $j$ and for every $t > 0$ an $N = N_j(\epsilon, t)$ such that

$$| [\psi_m(x - c_j{}^m)]_{2t};\ [\tau_j(x)]_{2t} | < \tfrac{1}{2}\epsilon \text{ for every } m > N_j(\epsilon, t), \tag{18}$$

if the number $c_j{}^m$ is suitably chosen. Notice that $c_j{}^m$ can be chosen as independent of $\epsilon$. On placing

$$M(\epsilon, t) = \operatorname{Max}\bigl(N_1(\epsilon, t), \cdots, N_j(\epsilon, t), \cdots, N_K(\epsilon, t)\bigr), \text{ where } K = K_\epsilon, \tag{19}$$

the statement (i) of Theorem 2 may be proved as follows:

Let $\rho$ be a given element of $\| \psi_k \|$. If $\rho(x)$ is of the form $\tau_j(x - b)$, where $j = 1, \cdots, K_\epsilon$ and $b$ is a number between $-t$ and $t$, then (i) is clear from (18) and (19). Suppose, therefore, that $\rho(x)$ is for no $j$ of the form $\tau_j(x - b)$, where $| b | \leqq t$. Then, on the one hand, there exists by Lemma 1 a $j$ such that

$$y_j \leqq \rho(x) \leqq y_{j+1} \text{ for } -t \leqq x \leqq t,$$

and, on the other hand, for this $j$,

$$y_j \leqq \tau_j(x + t) \leqq \rho(x) \text{ for } -t \leqq x \leqq t,$$

as seen from Lemma 1 and from (17). It follows, therefore, from (16) that, for this $j$,

$$| [\tau_j(x + t)]_t;\ [\rho(x)]_t | < \tfrac{1}{2}\epsilon;$$

cf. (II) and (IX). Since, by (18) and (19),

$$| [\psi_m(x + t - c_j{}^m)]_t;\ [\tau_j(x + t)]_t | < \tfrac{1}{2}\epsilon \text{ for } m > M(\epsilon, t),$$

it is seen from $(1_3)$ that

$$| [\psi_m(x + t - c_j{}^m)]_t;\ [\rho(x)]_t | < \tfrac{1}{2}\epsilon + \tfrac{1}{2}\epsilon \text{ for } m > M(\epsilon, t).$$

Hence, (i) in Theorem 2 is satisfied by $c = c_j{}^m - t$.

The converse statement of Theorem 2, namely (ii), is similarly proved.

**Convolutions.** If $\phi_1, \phi_2$ are two distribution functions, there exists a unique distribution function $\phi_1 * \phi_2$ which is defined by

$$(20) \qquad \phi_1(x) * \phi_2(x) = \int_{-\infty}^{+\infty} \phi_1(x-u)\,d\phi_2(u)$$

and is called the convolution of $\phi_1(x)$ and $\phi_2(x)$. It is known that

$$\phi_1 * \phi_2 = \phi_2 * \phi_1 \text{ and } (\phi_1 * \phi_2) * \phi_3 = \phi_1 * (\phi_2 * \phi_3).$$

It is seen from (20) that, for arbitrary numbers $c_1$, $c_2$,

$$(21) \quad \phi_1(x-c_1) * \phi_2(x-c_2) = \phi(x-c_1-c_2), \text{ where } \phi(x) = \phi_1(x) * \phi_2(x).$$

Use will be made also of the following facts:[8]

(XVI) If $\phi_n^1(x), \phi_n^2(x), \phi^1(x), \phi^2(x)$ are distribution functions such that $\phi_n^1 \rightarrow \phi^1$ and $\phi_n^2 \rightarrow \phi^2$, then $\phi_n^1 * \phi_n^2 \rightarrow \phi^1 * \phi^2$.

(XVII) If $\phi_n^1(x)$, $\phi_n^2(x)$, $\phi(x)$ are distribution functions such that $\phi_n^1 \rightarrow \phi$ and $\phi_n^1 * \phi_n^2 \rightarrow \phi$, then $\phi_n^2 \rightarrow \omega$, where $\omega(x) = \frac{1}{2}(1 + \operatorname{sgn} x)$.

A fact similar to (XVII) is

(XVIII) If $\phi_n^1(x)$, $\phi_n^2(x)$ are distribution functions such that $\phi_n^1 * \phi_n^2 \rightarrow \omega$, where $\omega(x) = \frac{1}{2}(1 + \operatorname{sgn} x)$, then there exists a sequence $\{c_n\}$ of numbers such that $\phi_n^1(x + c_n) \rightarrow \omega(x)$ and $\phi_n^2(x - c_n) \rightarrow \omega(x)$.

*Proof of* (XVIII). On placing $\omega_n = \phi_n^1 * \phi_n^2$, the assumption of (XVIII) is that, for every $\epsilon > 0$ and for some $N = N(\epsilon)$,

$$1 - \epsilon < \omega_n(\epsilon) - \omega_n(-\epsilon), \text{ if } n \geqq N(\epsilon).$$

Since $\omega_n = \phi_n^1 * \phi_n^2$,

$$\omega_n(\epsilon) - \omega_n(-\epsilon) = \int_{-\infty}^{+\infty} [\phi_n^1(\epsilon - u) - \phi_n^1(-\epsilon - u)]\,d\phi_n^2(u),$$

and so one can assume that there exists a $u$ for which

$$\omega_n(\epsilon) - \omega_n(-\epsilon) \leqq \phi_n^1(\epsilon - u) - \phi_n^1(-\epsilon - u).$$

Consequently, if $c_\epsilon^n$ denotes this $u$,

$$1 - \epsilon < \phi_n^1(\epsilon - c_\epsilon^n) - \phi_n^1(-\epsilon - c_\epsilon^n), \text{ if } n \geqq N(\epsilon).$$

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[8] B. Jessen and A. Wintner, *loc. cit.* [4], Section 3.

14

One can assume that $N(1/k) < N(1/(k+1))$, where $k = 1, 2, \cdots$. For a given $n > N(1)$, define an integer $k = k_n$ by the requirement that

$$N(1/k_n) < n \leqq N(1/k_{n+1}).$$

Then $k_n \to \infty$ as $n \to \infty$ and, for every $n$,

$$1 - k_n^{-1} < \phi_n^1(c_n + k_n^{-1}) - \phi_n^1(c_n - k_n^{-1}),$$

where $c_n$ denotes the negative value of $c_\epsilon^n$ for $\epsilon = k_n^{-1}$. Since $k_n^{-1} \to 0$ as $n \to \infty$, it follows that

$$\phi_n^1(x + c_n) \to \tfrac{1}{2}(1 + \operatorname{sgn} x) \text{ as } n \to \infty.$$

This, when combined with (21), clearly completes the proof of (XVIII).

**A lemma on convolutions.** The object of this section is the proof of a somewhat involved fact which isolates an essential part in the proof of Theorem 3 of the next section. The lemma in question (Lemma 3) is to the effect that if a convolution process $\phi_1 * \cdot$, when applied to a $\phi_2$, flattens $\phi_2$ strongly in the large, then also the local flattening of $\phi_2$ must be quite strong. Certain weaker results to the same effect have been indicated by Lévy.[5]

LEMMA 3. *Let $\lambda(x)$, $\mu(x)$ be two distribution functions such that there exist four positive constants $p$, $q$, $t$, $s$ which have the following properties:*

(i) $6q < p(1-p)$ *and* $p < 1$, *hence* $q < 1$;

(ii) $\lambda(x+2t) - \lambda(x) \leqq p + q$ *for every* $x$;

(iii) $\lambda(s) - \lambda(-s) \geqq 1 - q$;

(iv) $\mu(t) - \mu(-t) \geqq p - 2q$;

(v) $\mu(t+2s) - \mu(-t-2s) \leqq p + q$.

*Then there cannot exist a distribution function $\nu(x)$ such that $\lambda(x) * \nu(x) = \mu(x)$.*

*Proof.* Suppose, if possible, that there exists a distribution function $\nu(x)$ such that

$$\mu(x) = \int_{-\infty}^{+\infty} \lambda(x-u)\,d\nu(u).$$

Then, from (iv),

$$p - 2q \leqq \left\{ \int_{-\infty}^{-s-t} + \int_{-s-t}^{s+t} + \int_{s+t}^{+\infty} \right\} [\lambda(t-u) - \lambda(-t-u)]\,d\nu(u),$$

where, according to (ii),

$$\int_{-s-t}^{s+t} [\ ]d\nu(u) \leqq (p+q)[\nu(s+t)-\nu(-s-t)],$$

while, according to (iii),

$$\left\{\int_{-\infty}^{-s-t} + \int_{s+t}^{+\infty}\right\}[\ ]d\nu(u) \leqq q\int_{-\infty}^{+\infty} d\nu(u) = q,$$

so that

$$p-2q \leqq q + (p+q)[\nu(s+t)-\nu(-s-t)].$$

On the other hand, from (v) and from the assumption $\mu = \lambda * \nu$,

$$\begin{aligned} p+q &\geqq \mu(t+2s)-\mu(-t-2s) \\ &= \int_{-\infty}^{+\infty} [\lambda(2s+t-u)-\lambda(-2s-t-u)]d\nu(u), \end{aligned}$$

where $\int_{-\infty}^{+\infty} \geqq \int_{s+t}^{-s-t}$, and so, $\lambda(x)$ being a non-decreasing function which satisfies (iii),

$$p+q \geqq (1-q)[\nu(s+t)-\nu(-s-t)].$$

Consequently, by the inequality found before for $p-2q$,

$$p-2q \leqq q + (p+q)[(p+q)/(1-q)],$$

and so

$$p-p^2 \leqq 3q(1+p).$$

Since this contradicts (i), the proof of Lemma 3 is complete.

**Convolution sequences.** A sequence $\{\phi_n\}$ of distribution functions will be called a convolution sequence if there exists a sequence $\{\sigma_n\}$ of distribution functions such that

$$(22) \qquad \phi_n = \sigma_1 * \cdots * \sigma_n, \text{ i. e., } \phi_n = \phi_{n-1} * \sigma_n \qquad (\phi_1 = \sigma_1,\ n = 2, 3, \cdots).$$

(XIX) Every subsequence of a convolution sequence is a convolution sequence.

This is clear from $\chi_1 * (\chi_2 * \chi_3) = (\chi_1 * \chi_2) * \chi_3$.

(XX) If $\{\phi_n(x)\}$ is a convolution sequence, then so is $\{\phi_n(x-c_n)\}$ for every sequence of numbers $c_n$.

This is clear from (21).

THEOREM 3. *If, for a given convolution sequence $\{\phi_n\}$, one denotes by $\|\phi_n\|_0$ the function set obtained from $\|\phi_n\|$ by omitting the two constant functions $\alpha=0$, $\alpha=1$, then either every element of $\|\phi_n\|$ is a constant function or every element of $\|\phi_n\|_0$ is a distribution function. In the first case every constant function $\alpha$, where $0 \leqq \alpha \leqq 1$, is an element of $\|\phi_n\|$. In the second case there exists a distribution function $\rho$ such that a function is an element of $\|\phi_n\|_0$ if and only if it is congruent with $\rho$.* In the first case the convolution sequence $\{\phi_n\}$ will be said to be *flat*, in the second case *non-flat*.

The proof is somewhat lengthy and will be decomposed into the following steps ($\mathrm{XXI}_1$), ($\mathrm{XXI}_2$):

($\mathrm{XXI}_1$) If $\{\phi_n\}$ is a convolution sequence, then every non-constant function contained in the function set $\|\phi_n\|$ is a distribution function.

*Proof of* ($\mathrm{XXI}_1$). Let $\{\psi_m\}$ be a subsequence of $\{\phi_n\}$ such that $\{\psi_m\}$ has the properties described in Theorem 1. Thus there exists a number $p \geqq 0$ such that every $\rho \subset \|\psi_m\|$ has, for $-\infty < x < +\infty$, a total variation not greater than $p$, while the total variation of some $\rho_0 \subset \|\psi_m\|$ is equal to $p$. It is seen from (5) that ($\mathrm{XXI}_1$) will be proved if one shows that every non-constant function contained in $\|\psi_m\|$ is a distribution function. Suppose, if possible, that there exists in $\|\psi_m\|$ a non-constant function which is not a distribution function. Then, by the definition of $\rho_0$,

$$0 < p < 1, \text{ where } p = \rho_0(+\infty) - \rho_0(-\infty). \tag{23}$$

Hence one can choose a $q > 0$ for which condition (i) of Lemma 3 is satisfied. It is also seen from (23) that, for some $t > 0$,

$$\rho_0(t) - \rho_0(-t) > p - q. \tag{24}$$

On the other hand, by the definition of $p$,

$$p \geqq \rho(+\infty) - \rho(-\infty) \geqq \rho(x+2t) - \rho(x)$$

for every $\rho \subset \|\psi_m\|$ and for every $x$. Hence (i) of Theorem 2 assures the existence of a $\psi_k \subset \{\psi_m\}$ such that

$$\psi_k(x+2t) - \psi_k(x) \leqq p + q \text{ for every } x.$$

On denoting this $\psi_k$ by $\lambda$, condition (ii) of Lemma 3 is satisfied. Since $\lambda$ is a distribution function and $q > 0$, there clearly exists a number $s > 0$ which satisfies condition (iii) of Lemma 3. Since $\rho_0 \subset \| \psi_m \|$, there exists a sequence $\{c_m\}$ of numbers such that, at every continuity point $x$ of $\rho_0$,

$$\psi_m(x - c_m) \to \rho_0(x), \qquad m \to \infty.$$

Hence it is seen from (24) and (23) that there exists [9] a $\psi_l \subset \{\psi_m\}$ such that $l > k$ and

$$\psi_l(t - c_l) - \psi_l(-t - c_l) > p - q - q,$$
$$\psi_l(t + 2s - c_l) - \psi_l(-t - 2s - c_l) < p + q.$$

On denoting the distribution function $\psi_l(x - c_l)$ by $\mu(x)$, conditions (iv) and (v) of Lemma 3 are satisfied. Since $\{\psi_m\}$ is, by (XIX), a convolution sequence, and since

$$\lambda(x) = \psi_k(x),\ \mu(x) = \psi_l(x - c_l) \text{ and } l > k,$$

there exists, by (20) and (21), a distribution function $\nu(x)$ such that $\lambda * \nu = \mu$. Since this contradicts Lemma 3, the proof of ($\mathrm{XXI}_1$) is complete.

($\mathrm{XXI}_2$) If a distribution function $\rho$ is contained in $\| \psi_m \|$, where $\{\psi_m\}$ is a subsequence of a convolution sequence $\{\phi_n\}$, then $\rho$ is contained in $\| \phi_n \|$.

*Proof of* ($\mathrm{XXI}_2$). Suppose, if possible, that ($\mathrm{XXI}_2$) is false. Then ($\mathrm{XXI}_1$) implies that the set $\| \chi_k \|_0$ belonging to a suitably chosen subsequence $\{\chi_k\}$ of $\{\phi_n\}$ contains a function $\pi$ which is not congruent with $\rho$. In particular,

$$(25) \qquad \tau_k \to \pi, \qquad k \to \infty,$$

holds for a sequence of distribution functions $\tau_k(x)$ of the form $\chi_k(x - c_k)$. Since $\psi_m(x)$ and $\tau_k(x)$ are of the form $\phi_n(x - a_n)$, and since $\{\phi_n\}$ is a convolution sequence, there exist, by (21), for every $m$ two distribution functions $\lambda_m$, $\mu_m$ and two positive integers $k$, $j$ such that

$$(26) \qquad \psi_m * \lambda_m = \tau_k, \text{ where } k \to \infty \text{ as } m \to \infty,$$

and

$$(27) \qquad \tau_k * \mu_m = \psi_j, \text{ where } j > m.$$

Hence

$$(28) \qquad \psi_m * \lambda_m * \mu_m = \psi_j \text{ for some } j > m.$$

[9] The discontinuity points of $\rho_0$, if any, do not interfere with the possibility of this conclusion, since they form an at most enumerable set.

Since $\rho \subset \| \psi_m \|$, one can assume without loss of generality [cf. (IV) and (21)] that $\psi_m \rightarrow \rho$ as $m \rightarrow \infty$. Then (28) implies, in view of (XVII), that $\lambda_m * \mu_m \rightarrow \omega$. It follows, therefore, from (XVIII) that, for a suitably chosen sequence of numbers $b_m$,

$$(29) \qquad \lambda_m(x - b_m) \rightarrow \omega(x) \text{ as } m \rightarrow \infty.$$

Since, from (26) and (25),

$$(30) \qquad \psi_m(x) * \lambda_m(x) \rightarrow \pi(x),$$

and since $\pi(x)$ is not one of the constant functions equal to 0 or 1, it is easily inferred from (XVI) and (31) that the numbers $b_m$ tend to a limit $b$ as $m \rightarrow \infty$. Hence (29) can be replaced by

$$(31) \qquad \lambda_m(x) \rightarrow \omega(x + b).$$

It follows, therefore, from (30) (XVI) and from the definition $\psi_m(x) \rightarrow \rho(x)$ of $\rho$ that

$$(32) \qquad \rho(x) * \omega(x + b) = \pi(x).$$

This means in view of the definition of $\omega(x)$ that $\rho(x + b) = \pi(x)$, i. e., that $\pi$ is a distribution function and that $\rho$ and $\pi$ are congruent. Since this contradicts the assumption, the proof of $(XXI_2)$ is complete.

*Proof of Theorem* 3. Suppose that $\| \phi_n \|$ does not contain all constant functions $\alpha$, where $0 \leqq \alpha \leqq 1$. Then $\{\phi_n\}$ contains, by (XV), a subsequence $\{\psi_m\}$ such that $\| \psi_m \|$ contains a non-constant function, say $\rho$. This $\rho$ is, by $(XXI_1)$, a distribution function. Furthermore, $\rho \subset \| \phi_n \|$, by $(XXI_2)$. Finally, it is seen from (6) and Lemma 1 that a function is contained in $\| \phi_n \|_0$ if and only if it is congruent with $\rho$. This completes the proof of Theorem 3.

There arises the question how to decide whether or not a given convolution sequence $\{\phi_n\}$ is flat in the sense of Theorem 3. In order to obtain criteria to this effect, put

$$(33_1) \qquad E(\chi) = \int_{-\infty}^{+\infty} x d\chi(x),$$

if the distribution function $\chi$ has a first moment, which is certainly the case if $\chi$ has a finite second moment

$$(33_2) \qquad F(\chi) = \int_{-\infty}^{+\infty} x^2 d\chi(x).$$

In the latter case, let

(33) $$D(\chi) = F(\chi) - [E(\chi)]^2.$$

Thus $D(\chi) = \int_{-\infty}^{+\infty} [x - E(\chi)]^2 d\chi(x)$; hence

(34) $$D(\chi) \geqq 0,$$

where $D(\chi) = 0$ if and only if $\chi$ is congruent with $\omega = \frac{1}{2}(1 + \operatorname{sgn} x)$. If $F(\chi) = +\infty$, put $D(\chi) = +\infty$.

It is clear from ($33_1$) that

($35_1$) $$E(\chi_1) = E(\chi_2) + c, \text{ if } \chi_1(x) = \chi_2(x - c).$$

Similarly, from ($33_2$),

($35_2$) $$F(\chi_1) = F(\chi_2) + 2cE(\chi_2) + c^2, \text{ if } \chi_1(x) = \chi_2(x - c).$$

Consequently, from (33),

(35) $$D(\chi_1) = D(\chi_2), \text{ if } \chi_1 \text{ and } \chi_2 \text{ are congruent.}$$

Hence, on combining (IV), (21) and Theorem 3 with known [10] criteria for the convergence of infinite convolutions, it is seen that Theorem 3 can be completed by

THEOREM 4. *In order that a convolution sequence* $\{\phi_n\}$ *be non-flat, it is sufficient that, on using the notations* (33) *and* (22),

(36) $$\sum_{n=1}^{\infty} D(\sigma_m) < +\infty \qquad \text{[cf. (34)]}.$$

*This sufficient condition is necessary as well in case there exists a sufficiently large* $L > 0$ *such that the spectrum of every* $\sigma_n$ *is a subset of an x-interval of suitably chosen position and of length L, where L is independent of n.*

**Convergent infinite convolutions.** Let $\{\phi_n\}$ be a convolution sequence, i. e., a sequence of distribution functions which can be represented in the form (22). It is clear that $\{\phi_n\}$ can converge, in the sense of ($3_1$), to a $\phi$ which is not a distribution function. The infinite convolution $\phi = \sigma_1 * \sigma_2 * \cdots$ is said to be a convergent infinite convolution only if (22) satisfies ($3_1$) with a function $\phi$ which is a distribution function. On using this terminology, Theorem 3 clearly implies

[10] B. Jessen and A. Wintner, *loc. cit.* [4], Theorem 4 and Theorem 5.

THEOREM 5. *If a convolution sequence*

$$\{\phi_n\} = \{\sigma_1 * \cdots * \sigma_n\} \tag{37}$$

*tends, in the sense of* $(3_1)$, *to a limit function* $\phi$, *and if the infinite convolution* $\phi = \sigma_1 * \sigma_2 * \cdots$ *is not convergent, then* $\phi$ *is a constant function.*

Theorem 3 also implies [11]

THEOREM 6. *There exists for every non-flat convolution sequence* (37) *a sequence* $\{c_n\}$ *of numbers for which the infinite convolution*

$$\sigma_1(x - c_1) * \sigma_2(x - c_2) * \cdots \tag{38}$$

*is convergent. If* $\{c_n^1\}$ *and* $\{c_n^2\}$ *are two such sequences* $\{c_n\}$, *then the two corresponding infinite convolutions* (38) *represent congruent distribution functions.*

*If, on the other hand, a convolution sequence* (37) *is flat, then the infinite convolution* (38) *is divergent for every* $\{c_n\}$.

In what follows, use will be made of the following fact which is merely a restatement [12] of a part of the fundamental result of Khintchine and Kolmogoroff [2] concerning "equivalent" series of independent random variables:

(XXII) The infinite convolution $\sigma_1 * \sigma_2 * \cdots$ is convergent if and only if so is the infinite convolution $[\sigma_1]_t * [\sigma_2]_t * \cdots$ for a $t > 0$ (in which case the same holds for every $t > 0$). It is understood that $[\sigma]_t$ denotes the distribution function defined in (IX).

On combining (XXII) with the known convergence criterion [10] for a convolution of distribution functions with uniformly bounded spectra, one obtains

(XXIII) The infinite convolution $\sigma_1 * \sigma_2 * \cdots$ is convergent if and only if so are both series

$$\sum_{n=1}^{\infty} E([\sigma_n]_t), \qquad \sum_{n=1}^{\infty} D([\sigma_n]_t) \qquad \text{[cf. } (33_1),\ (33)]$$

for a $t > 0$ (in which case the same holds for every $t > 0$).

**Absolutely convergent infinite convolutions.** Two infinite convolutions, $\sigma_1' * \sigma_2' * \cdots$ and $\sigma_1'' * \sigma_2'' * \cdots$, will be said to be rearrangements of each

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[11] This theorem has been stated without a detailed proof by P. Lévy, *loc. cit.* [5], p. 340; cf. also p. 337.

[12] Cf. B. Jessen and A. Wintner, *loc. cit.* [4], Theorems 32 and 34.

other if the sequences $\{\sigma_n'\}$ and $\{\sigma_n''\}$ are permutations of each other. An infinite convolution $\sigma_1 * \sigma_2 * \cdots$ is said to be absolutely convergent if every rearrangement of it is a convergent infinite convolution. It is known [8] that in this case all rearrangements represent the same distribution function. The definition of an absolutely convergent infinite convolution clearly implies

(XXIV) Both (XXII) and (XXIII) remain valid if one reads "absolutely convergent" instead of "convergent."

THEOREM 7. *There exists for every convergent infinite convolution $\sigma_1(x) * \sigma_2(x) * \cdots$ a sequence $\{c_n\}$ of numbers such that the infinite convolution $\sigma_1(x - c_1) * \sigma_2(x - c_2) * \cdots$ is absolutely convergent.* Needless to say, another sequence, $\{\bar{c}_n\}$, of numbers has this property if and only if $| c_1 - \bar{c}_1 | + | c_2 - \bar{c}_2 | + \cdots$ is a convergent series.

*Proof.* Choose a fixed $t > 0$, put

$$(39) \qquad \rho_n(x) = [\sigma_n(x)]_{2t},$$

where $[\ \ ]_{2t}$ is defined by (IX), and let

$$(40_1) \quad \tau_n(x) = \rho_n(x - c_n); \qquad (40_2) \quad \chi_n(x) = \sigma_n(x - c_n),$$

where, on using the notation $(33_1)$, the number $c_n$ is chosen as follows:

$$(41) \quad c_n = 0 \text{ or } c_n = E(\rho_n) \text{ according as } | E(\rho_n) | > t \text{ or } | E(\rho_n) | \leqq t.$$

Thus $| c_n | \leqq t$, and so (39), $(40_1)$, $(40_2)$ and (IX) imply that

$$[\chi_n]_t = [\tau_n]_t.$$

Hence (XXIV) and $(40_2)$ show that the infinite convolution (38) is absolutely convergent if and only if so is the infinite convolution $\tau_1(x) * \tau_2(x) * \cdots$, i. e., if and only if so are both series

$$(42_1) \quad \sum_{n=1}^{\infty} E(\tau_n); \qquad (42_2) \quad \sum_{n=1}^{\infty} D(\tau_n).$$

Now $(42_1)$ is absolutely convergent, since, on the one hand,

$$E(\tau_n) = E(\rho_n) - c_n, \text{ by } (35_1) \text{ and } (40_1),$$

and, on the other hand,

$$E(\rho_n) = c_n \text{ for every sufficiently large } n,$$

as seen from (41) and from the fact that $\sum_{n=1}^{\infty} E(\rho_n)$ is, by (XXIII), (39) and the assumption of Theorem 7, a convergent series. On using (35) instead of ($35_1$) and (34) instead of (41), it is similarly shown that ($42_2$) is absolutely convergent. Consequently, the infinite convolution (38) is absolutely convergent. This completes the proof of Theorem 7.

Theorem 7 implies, in view of Theorem 6,

THEOREM 8. *A convolution sequence* (37) *is non-flat if and only if there exists a sequence* $\{c_n\}$ *of numbers such that the infinite convolution* (38) *is absolutely convergent.*[11]

Theorem 8 and Theorem 3 imply

THEOREM 9. *If two sequences* $\{\sigma_n'\}$, $\{\sigma_n''\}$ *of distribution functions are permutations of each other, then the two function sets* $\| \phi_n' \|_0$, $\| \phi_n'' \|_0$, *where* $\phi_n' = \sigma_1' * \cdots * \sigma_n'$, $\phi_n'' = \sigma_1'' * \cdots * \sigma_n''$, *are identical function sets. In other words, either both convolution sequences* $\{\phi_n'\}$, $\{\phi_n''\}$ *are flat or both are non-flat, and in the latter case the two distribution functions* $\rho'$, $\rho''$, *which* $\| \phi_n' \|_0$, $\| \phi_n'' \|_0$ *determine up to congruences, are congruent.*

Theorems 3 and 8 can be interpreted as describing the possible behavior of any divergent infinite convolution. Correspondingly, Theorem 9 describes what can happen to a non-absolutely convergent infinite convolution upon an arbitrary rearrangement of its "factors." In the non-flat case, Theorems 8 and 9 imply the more precise fact that *the infinite convolution behaves upon a rearrangement exactly the same way as a certain numerical series,* which is determined by the sequence of the "factors" up to an additive absolutely convergent numerical series.

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# AN ANALOGUE OF JACOBI'S CONDITION FOR THE PROBLEM OF MAYER WITH VARIABLE END POINTS.*

By Thomas Freeman Cope.

**Introduction.** The problem of Mayer with variable end points, as stated by Bliss,[1] is the determination of the properties of an arc

$$\text{(E)} \qquad y_i = y_i(x), \qquad x_1 \leqq x \leqq x_2, \qquad (i = 1, \cdots, n),$$

which minimizes the first of a set of functions

$$f_\rho[x_1, x_2, y(x_1), y(x_2)], \qquad (\rho = 1, \cdots, r \leqq 2n + 2),$$

in the class of similar arcs which make $f_2, \cdots, f_r$ vanish and besides satisfy the differential equations

$$(1) \qquad \phi_\alpha(x, y, y') = 0, \qquad (\alpha = 1, \cdots, m < n),$$

where $y(x)$ is a symbol for $y_1 \cdots y_n$. Bliss has shown[2] that if $\Omega$ denotes the function

$$\Omega = \lambda_1\phi_1 + \cdots + \lambda_m\phi_m$$

then a necessary condition for a minimum is that there shall exist $m$ functions $\lambda_1(x), \cdots, \lambda_m(x)$, not all identically zero on $x_1x_2$, satisfying the differential equations

$$(2) \qquad \Omega_{y_i} - \frac{d}{dx}\Omega_{y'_i} = 0, \qquad (i = 1, \cdots, n),$$

and making all determinants of order $r + 1$ of the matrix

$$(3) \qquad \left\| \begin{matrix} f_{\rho x_1} & f_{\rho y_{i1}} & f_{\rho x_2} & f_{\rho y_{i2}} \\ \Omega(x_1) - \Omega_{y'_i}(x_1)y'_{i1}, & \Omega_{y'_i}(x_1), & -\Omega(x_2) + \Omega_{y'_i}(x_2)y'_{i2}, & -\Omega_{y'_i}(x_2) \end{matrix} \right\|$$

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* Presented to the American Mathematical Society (Chicago), April, 1928. Received by the Editors March 1, 1937. This paper is a somewhat revised form of the author's dissertation, see 4. The great interest shown in the problem and methods of this paper, as attested by the bibliography at the end, seemed to the author to justify its publication at this time.

The numbers in the footnotes refer to the bibliography at the end where further references will be found.

[1] 1. [2] 1.

vanish. According to the usual convention of tensor analysis, it is understood here and elsewhere in this paper that a subscript $i$, $j$, $\alpha$, $\beta$, etc., repeated in the same term indicates a sum. The subscripts $x$, $y$, $y'$ denote partial derivatives. It is also understood that the arguments in $f_\rho$, $\Omega$, and their derivatives are those belonging to E.

The preceding statement of the necessary condition is equivalent to the statement [3] that there must exist $m$ functions $\lambda_\alpha(x)$, not all identically zero on $x_1x_2$, satisfying the equations (2), and $r$ constants $l_1, \cdots, l_r$, not all zero, satisfying with them the $2n + 2$ equations

$$(4) \quad f_{x_1} + f_{y_{i1}} y'_{i1} = f_{y_{i1}} - \Omega_{y'_i}(x_1) = -f_{x_2} - f_{y_{i2}} y'_{i2} = -f_{y_{i2}} - \Omega_{y'_i}(x_2) = 0,$$

where

$$f = l_1 f_1 + \cdots + l_r f_r.$$

In the first section of this paper, the hypotheses on which the analysis is based, are stated and preliminary notions and theorems considered. The first and second variations are computed in section 2. It is shown in section 3 that a minimum problem for the second variation may be formulated and moreover that it can be transformed into a problem of the same type as the original one. The differential equations and boundary conditions corresponding to (2) and (4) above for this auxiliary minimum problem are then given. In section 4 a boundary value problem associated with the second variation is stated and discussed, and by means of it a necessary condition for the original minimum problem is proved. This condition is essentially that for a minimizing arc for the original problem the boundary value problem of the second variation can have no solution for negative values of its parameter. It is then shown in section 5 that the boundary value problem of the second variation can be transformed into one that has been treated by Bliss. Section 6 is devoted to the task of proving that the transformed boundary value problem is " definitely self-adjoint," according to Bliss's definition. From this fact much information is automatically obtained about the characteristic constants and solutions of the original boundary value problem.

The form of the second variation appearing in section 2, the analogue of the Jacobi necessary condition of section 4, and the discussion of the boundary value problem of the second variation of sections 5 and 6 were first given, for the general problem of Mayer with variable end points, in my dissertation (see 4). Proofs of similar analogues have since been published by Morse (see 7, p. 524) and Reid (see 14, p. 840). The same authors have also

[3] 1, p. 311.

treated the boundary value problem of the second variation by methods quite different from mine and from one another (see 7, pp. 542-546, and 10). Analogues of the Jacobi necessary condition different in form from those mentioned above have been given by Bliss (see 8, p. 266) and Hestenes (see 12, p. 483).

**1. Preliminary notions and theorems.** The arc $E$ is supposed to have the following properties:[4]

1. It is of class $C'''$ and such that the functions $\phi_\alpha$, $f_\rho$ are of class $C^{\mathrm{IV}}$ in a neighborhood $R$ of the values $(x, y, y')$ on $E$.

2. It satisfies the equations (1) and

$$f_\sigma = 0, \qquad (\sigma = 2, \cdots, r). \tag{5}$$

3. The matrix $\| \phi_{\alpha y'_i} \|$ has rank $m$ at every point of $E$.

4. The matrix

$$\| f_{\rho x_1} f_{\rho y_{i1}} f_{\rho x_2} f_{\rho y_{i2}} \|, \qquad (\rho = 1, \cdots, r; i = 1, \cdots, n),$$

with $2n + 2$ columns and $r$ rows is of rank $r$ at the values of the arguments of the functions $f_\rho$ on $E$.

Consider a one-parameter family of arcs[5]

$$\begin{aligned} y_i &= Y_i(x, \epsilon), && (i = 1, \cdots, n), \\ X_1(\epsilon) \leqq x &\leqq X_2(\epsilon), \quad X_\iota(0) = x_\iota, && (\iota = 1, 2), \end{aligned} \tag{6}$$

which contains $E$ for $\epsilon = 0$ and satisfies the equations (1) and (5) for every $\epsilon$ in a neighborhood of $\epsilon = 0$. Its variations are by definition the expressions

$$\xi_\iota = X_{\iota\epsilon}(0), \qquad \eta_i(x) = Y_{i\epsilon}(x, 0).$$

The variations then satisfy the equations of variation

$$\Phi_\alpha(x, \eta, \eta') \equiv \phi_{\alpha y_i}\eta_i + \phi_{\alpha y'_i}\eta'_i = 0, \qquad (\alpha = 1, \cdots, m), \tag{7}$$

$$F_\sigma(\xi, \eta) = 0, \qquad (\sigma = 2, \cdots, r), \tag{8}$$

where

$$\begin{aligned} F_\rho(\xi, \eta) \equiv {} & (f_{\rho x_1} + f_{\rho y_{i1}} y'_{i1})\xi_1 + f_{\rho y_{i1}}\eta_{i1} \\ & + (f_{\rho x_2} + f_{\rho y_{i2}} y'_{i2})\xi_2 + f_{\rho y_{i2}}\eta_{i2}, \qquad (\rho = 1, \cdots, r). \end{aligned} \tag{9}$$

The functions $y$, $y'$ occurring explicitly and in the derivatives are those defining $E$.

Consider now a system $H$ of $r$ sets of variations $\xi_\iota^\rho$, $\eta_i^\rho$ $(\rho = 1, \cdots, r;$

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[4] 1, p. 307.

[5] 1, p. 307.

$\iota = 1, 2$), with $\eta$'s of class $C'''$ and satisfying the equations (7). Variations of this sort with $\xi_1$ and $\xi_2$ arbitrary constants are called *admissible* variations. A minimizing arc $E$ is said to be normal for the problem under consideration when a system $H$ of variations can be so selected that the matrix

$$\| F_\sigma(\xi^\rho, \eta^\rho) \|, \qquad (\sigma = 2, \cdots, r;\ \rho = 1, \cdots, r)$$

has rank $r - 1$. Let an admissible arc be defined as an arc of class $C'''$ on $x_1x_2$, whose elements $(x, y, y')$ all lie in $R$, and which satisfies the equations (1). The following theorem and its corollary with their proofs, which are omitted here, are similar to those given by Bliss in his lectures at Chicago in the summer of 1925.[6]

THEOREM. *For every normal minimizing arc $E$ of class $C'''$ on $x_1x_2$ for the Mayer problem with variable end points, there exists a one-parameter family of admissible arcs* (6) *containing $E$ for $\epsilon = 0$ and satisfying the equations* (5). *The functions $Y_i$ are of class $C'''$ in $x$, and $Y_i$, $Y'_i$, $X_\iota$ $(i = 1, \cdots, n;\ \iota = 1, 2)$ of class $C''$ in $\epsilon$, near $x_1 \leqq x \leqq x_2$, $\epsilon = 0$.*

COROLLARY. *If a set of admissible variations $\xi$, $\eta(x)$ for a normal minimizing arc $E$ of class $C'''$ on $x_1x_2$ for the Mayer problem with variable end points satisfies the equations* (8), *there exists a one-parameter family of admissible arcs* (6) *satisfying the end conditions* (5), *containing the arc $E$ for $\epsilon = 0$, and having the set $\xi$, $\eta$ as its variations along $E$. The functions $Y_i$ are of class $C'''$ in $x$ and $Y_i$, $Y'_i$, $X_\iota$ of class $C''$ in $\epsilon$ near $x_1 \leqq x \leqq x_2$, $\epsilon = 0$.*

**2. The first and second variations.** Consider the minimizing arc $E$ of the corollary. There must exist $m$ functions $\lambda_\alpha(x)$ of class $C'$, not all identically zero on $x_1x_2$, satisfying the equations (2), and $r$ constants $l_\rho$, not all zero, satisfying the equations (4). Moreover, since $E$ is normal, $l_1$ must be different from zero.[7] $l_1$ may then be chosen equal to unity, and in the following pages it will be assumed that such a choice of $l_1$ has been made.

Substitute the one-parameter family of the corollary in the functions $f_\rho$, $\phi_\alpha$ and differentiate with respect to $\epsilon$. The result is

$$\begin{aligned} \frac{df_1}{d\epsilon} &= F_1(X_\epsilon, Y_\epsilon), && \\ 0 &= F_\sigma(X_\epsilon, Y_\epsilon), && (\sigma = 2, \cdots, r), \\ 0 &= \Phi_\alpha(x, Y_\epsilon, Y'_\epsilon), && (\alpha = 1, \cdots, m), \end{aligned}$$

[6] 5, pp. 694-695, and 4, pp. 6-9.

[7] 1, p. 311.

where $F_\rho$ and $\Phi_\alpha$ with the arguments indicated are defined by equations (7) and (9), but with coefficients taken for $\epsilon = \epsilon$. Differentiating again with respect to $\epsilon$ and putting $\epsilon = 0$, we find

$$
\begin{aligned}
&\left(\frac{d^2 f_1}{d\epsilon^2}\right)\Bigg|^{\epsilon=0} &&= F_1(X_{\epsilon\epsilon}, Y_{\epsilon\epsilon}) + 2f_{1y_j}Y'_{j\epsilon}X_\epsilon\,|^1 + 2f_{1y_j}Y'_{j\epsilon}X_\epsilon\,|^2 + Q_1(X_\epsilon, Y_\epsilon),\\
(10)\quad &0 &&= F_\sigma(X_{\epsilon\epsilon}, Y_{\epsilon\epsilon}) + 2f_{\sigma y_j}Y'_{j\epsilon}X_\epsilon\,|^1 + 2f_{\sigma y_j}Y'_{j\epsilon}X_\epsilon\,|^2 + Q_\sigma(X_\epsilon, Y_\epsilon),\\
&0 &&= \Phi_\alpha(x, Y_{\epsilon\epsilon}, Y'_{\epsilon\epsilon}) + 2\omega_\alpha(x, Y_\epsilon, Y'_\epsilon),\\
& &&\qquad (j = 1, \cdots, n;\ \sigma = 2, \cdots, r;\ \alpha = 1, \cdots, m),
\end{aligned}
$$

where $Q_1$, $Q_\sigma$, $2\omega_\alpha$ are quadratic forms in the arguments indicated, and where all the arguments are taken for $\epsilon = 0$. Multiply the first $r$ equations of (10) by $l_1$, $l_\sigma$, respectively, and add. Then because of the equations (4) of section 1, the sum can be written

$$
(11)\qquad \left(\frac{d^2 f_1}{d\epsilon^2}\right)\Bigg|^{\epsilon=0} = \Omega_{y'_j}Y_{j\epsilon\epsilon}\Bigg|_{X_2(\epsilon)}^{X_1(\epsilon)} + 2\Omega_{y'_j}Y'_{j\epsilon}X_\epsilon\Bigg|_{X_2(\epsilon)}^{X_1(\epsilon)} + l_\rho Q_\rho(X_\epsilon, Y_\epsilon),\quad (\rho = 1, \cdots, r),
$$

where all the elements are taken for $\epsilon = 0$. Multiply now the last $m$ equations of (10) by $\lambda_\alpha$ and add. The result is, if we put $\omega = \lambda_\alpha \omega_\alpha$,

$$
\begin{aligned}
0 &= \Omega_{y_j}Y_{j\epsilon\epsilon} + \Omega_{y'_j}Y'_{j\epsilon\epsilon} + 2\omega(x, Y_\epsilon, Y'_\epsilon)\\
&= Y_{j\epsilon\epsilon}\left(\Omega_{y_j} - \frac{d}{dx}\Omega_{y'_j}\right) + \frac{d}{dx}(\Omega_{y'_j}Y_{j\epsilon\epsilon}) + 2\omega\\
&= \frac{d}{dx}(\Omega_{y'_j}Y_{j\epsilon\epsilon}) + 2\omega,
\end{aligned}
$$

since for $\epsilon = 0$, the Euler-Lagrange equations (2) are true. Integrating between $X_1(\epsilon)$ and $X_2(\epsilon)$ for $\epsilon = 0$, we obtain

$$
(12)\qquad 0 = \Omega_{y'_j}Y_{j\epsilon\epsilon}\Bigg|_{X_1(\epsilon)}^{X_2(\epsilon)} + \int_{X_1(\epsilon)}^{X_2(\epsilon)} 2\omega(x, Y_\epsilon, Y'_\epsilon)\,dx.
$$

Finally by multiplying the equations

$$0 = \Phi_\alpha(x, Y_\epsilon, Y'_\epsilon)$$

by $\lambda_\alpha$ and adding, we find

$$0 = \Omega_{y_j}Y_{j\epsilon} + \Omega_{y'_j}Y'_{j\epsilon},$$

whence

$$
(13)\qquad 0 = 2\Omega_{y_j}Y_{j\epsilon}X_\epsilon\Bigg|_{X_1(\epsilon)}^{X_2(\epsilon)} + 2\Omega_{y'_j}Y'_{j\epsilon}X_\epsilon\Bigg|_{X_1(\epsilon)}^{X_2(\epsilon)},
$$

where as before we take $\epsilon = 0$. Now add (11), (12), and (13). The result will be the desired form of the second variation, after putting

$$X_{1\epsilon}(0)=\xi_1,\qquad X_{2\epsilon}(0)=\xi_2,\qquad Y_j(x,\epsilon)\,|^{\epsilon=0}=\eta_j(x),$$

namely,

$$I_2\equiv\left(\frac{d^2f_1}{d\epsilon^2}\right)\Big|^{\epsilon=0}=\int_{x_1}^{x_2}2\omega(x,\eta,\eta')\,dx+Q(\xi,\eta),\tag{14}$$

where the quadratic forms $2\omega$ and $Q$ are explicitly

$$\begin{aligned}
2\omega &= P_{ij}\eta_i\eta_j+2Q_{ij}\eta_i\eta'_j+R_{ij}\eta'_i\eta'_j;\\
Q &= A\xi_1^2+2B\xi_1\xi_2+C\xi_2^2+2D_j\eta_j(x_1)\xi_1+2E_j\eta_j(x_2)\xi_1\\
&\quad+2F_j\eta_j(x_1)\xi_2+2G_j\eta_j(x_2)\xi_2+H_{ij}\eta_i(x_1)\eta_j(x_1)\\
&\quad+2I_{ij}\eta_i(x_1)\eta_j(x_2)+J_{ij}\eta_i(x_2)\eta_j(x_2);
\end{aligned}$$

$$\begin{aligned}
&P_{ij}=\Omega_{y_iy_j},\qquad Q_{ij}=\Omega_{y_iy'_j},\qquad R_{ij}=\Omega_{y'_iy'_j};\\
&A=\frac{d}{dx_1}\left(f_{x_1}+f_{y_{j1}}y'_j(x_1)\right),\qquad B=\frac{d}{dx_1}\left(f_{x_2}+f_{y_{j2}}y'_j(x_2)\right),\\
&C=\frac{d}{dx_2}\left(f_{x_2}+f_{y_{j2}}y'_j(x_2)\right),\qquad D_j=\frac{d}{dx_1}f_{y_{j1}}-\Omega_{y_j}(x_1),\\
&E_j=\frac{d}{dx_1}f_{y_{j2}},\qquad F_j=\frac{d}{dx_2}f_{y_{j1}},\qquad G_j=\frac{d}{dx_2}f_{y_{j2}}+\Omega_{y_j}(x_2),\\
&H_{ij}=f_{y_{i1}y_{j1}},\qquad I_{ij}=f_{y_{i1}y_{j2}},\qquad J_{ij}=f_{y_{i2}y_{j2}},\\
&\qquad\qquad(i,j=1,\cdots,n).
\end{aligned}\tag{15}$$

If $y_i=y_i(x)$ is a minimizing arc it is evidently necessary that the first variation $I_1$,

$$I_1\equiv\left(\frac{df_1}{d\epsilon}\right)\Big|^{\epsilon=0}=F_1(\xi,\eta),$$

vanish for all sets $\xi$, $\eta$ satisfying the differential equations and end conditions (7) and (8). It is also necessary that the second variation $I_2$ be greater than or equal to zero for the same sets $\xi$, $\eta$.

**3. The minimum problem of the second variation.** A problem suggesting itself at this point is that of minimizing the second variation (14) in the class of all sets of variations $\xi$, $\eta$ of class $C'''$ on $x_1x_2$, satisfying the equations

$$\begin{aligned}
&\frac{d\xi_\iota}{dx}=0,\qquad \Phi_\alpha(x,\eta,\eta')=0,\qquad (\iota=1,2;\ \alpha=1,\cdots,m),\\
&\qquad F_\sigma(\xi,\eta)=0,\qquad\qquad (\sigma=2,\cdots,r).
\end{aligned}\tag{16}$$

It is evident that the second variation $I_2$ must be greater than or equal to zero in the class of all such sets $\xi,\eta$. It is found convenient to put a further restriction on the sets $\xi,\eta$. Let us introduce the equation

$$\xi_1^2+\xi_2^2+\int_{x_1}^{x_2}\eta_i(x)\eta_i(x)\,dx=1.\tag{17}$$

We then consider a second problem of the second variation and its relation to

the one first proposed. This second problem is to minimize the second variation in the class of all sets $\xi, \eta$ satisfying the equations (16) and (17). We shall call $\xi, \eta$ an admissible set for the first problem if the $\xi$'s and $\eta$'s are of class $C'''$ on $x_1x_2$, and if the set satisfies the equations (16); and an admissible set for the second problem if the set further satisfies the equation (17). Every admissible set for the first problem, except the trivial one $\xi = \eta = 0$, will evidently give rise to a set $k\xi, k\eta$, which is an admissible set for the second problem, where $k$ is a constant different from zero. Moreover, every admissible set $\xi, \eta$ of the second problem is necessarily an admissible set of the first problem. It follows that the admissible sets of the second problem form a subclass of the class of all admissible sets of the first problem. We can then state the following

THEOREM. *If the second variation $I_2(\xi, \eta)$ is greater than or equal to zero for all sets $\xi, \eta$ which satisfy the equations* (16), *where the $\xi$'s and $\eta$'s are of class $C'''$ on $x_1x_2$, then $I_2(\xi, \eta)$ is necessarily greater than or equal to zero for all such sets $\xi, \eta$ which satisfy both the equations* (16) *and* (17) *and conversely.*

The problem that will now be considered is that of minimizing the second variation $I_2(\xi, \eta)$ in the class of all sets $\xi, \eta$ which satisfy the equations (16) and (17) with $\xi$'s and $\eta$'s of class $C'''$ on $x_1x_2$. This problem is not quite in the form of the original problem in $xy$-space, but may be changed to that form by the introduction of new variables. For, let $\eta_0(x)$, $\eta_{n+1}(x)$ be defined by

$$\eta_0(x) = \int_{x_1}^{x} 2\omega(x, \eta, \eta')\,dx, \quad \eta_{n+1}(x) = \int_{x_1}^{x} \eta_i(x)\eta_i(x)\,dx,$$

$$\eta_0(x_1) = 0,\ \eta_0(x_2) = \int_{x_1}^{x_2} 2\omega(x, \eta, \eta')\,dx,\ \eta_{n+1}(x_1) = 0,\ \eta_{n+1}(x_2) = \int_{x_1}^{x_2} \eta_i\eta_i\,dx.$$

Let also the values of the constants $\xi_1, \xi_2$ at the end points $x_1$ and $x_2$ be denoted by $\xi_{11}, \xi_{21}$ and $\xi_{12}, \xi_{22}$ respectively, and consider (as we obviously may with no loss of generality) only $\xi_{11}$ and $\xi_{22}$ as occurring in the equations (15) and (16) and also in the second variation. We can then formulate an auxiliary Mayer problem as follows:

To find the properties of a set of functions

$$\eta_0(x),\ \eta_i(x),\ \eta_{n+1}(x),\ \xi_\iota(x), \qquad (i = 1, \cdots, \eta;\ \iota = 1, 2),$$

which minimizes the expression

$$g_1 \equiv I_2 = \eta_0(x_2) + Q(\xi_{11}, \xi_{22}, \eta_i(x_1), \eta_i(x_2)),$$

in the class of such sets of functions satisfying the differential equations

$$(18)\quad \begin{gathered} \eta'_0 - 2\omega(x,\eta,\eta') = 0, \qquad \Phi_\alpha(x,\eta,\eta') \equiv \phi_{\alpha y_j}\eta_j + \phi_{\alpha y'_j}\eta'_j = 0, \\ \eta'_{n+1} - \eta_i\eta_i = 0, \qquad \xi'_\iota = 0, \\ (i, j = 1,\cdots,n;\ \iota = 1, 2;\ \alpha = 1,\cdots,m), \end{gathered}$$

and the end conditions,

$$(19)\quad \begin{aligned} & g_2 \equiv \eta_0(x_1) = 0, \\ & g_{\sigma+1} \equiv F_\sigma(\xi_{11}, \xi_{22}, \eta_i(x_1), \eta_i(x_2)) = 0, \qquad (\sigma = 2,\cdots,r), \\ & g_{r+2} \equiv \eta_{n+1}(x_1) = 0, \\ & g_{r+3} \equiv \xi^2_{11} + \xi^2_{22} + \eta_{n+1}(x_2) - 1 = 0, \end{aligned}$$

with $\xi$'s and $\eta$'s of class $C'''$ on $x_1x_2$.

This auxiliary problem is a Mayer problem of the type considered in the introduction and so a minimizing set of functions

$$\eta_0(x),\ \eta_i(x),\ \eta_{n+1}(x),\ \xi_\iota(x), \qquad (i = 1,\cdots,n;\ \iota = 1, 2),$$

of class $C'''$ on $x_1x_2$ must satisfy the conditions there stated.[8] Let $\mu_0$ be the multiplier associated with $\eta'_0 - 2\omega$; $\mu_\alpha(x)$, $(\alpha = 1,\cdots,m)$, those associated with $\Phi_\alpha$; $\mu_{m+1}$, $\mu_{m+2}$, those associated with $\xi'_1$, $\xi'_2$, respectively; and, finally, $\mu$, that associated with $\eta'_{n+1} - \eta_i\eta_i$, where the $\mu$'s are of class $C'$ on $x_1x_2$. Define $\Gamma$ by the equation

$$2\Gamma \equiv \mu_0(\eta'_0 - 2\omega) + 2\mu_\alpha\Phi_\alpha + 2\mu_{m+1}\xi'_1 + 2\mu_{m+2}\xi'_2 + \mu(\eta'_{n+1} - \eta_i\eta_i).$$

Then the Euler differential equations for the auxiliary problem are

$$\frac{d}{dx}\Gamma_{\eta'_0} = \Gamma_{\eta_0},\ \frac{d}{dx}\Gamma_{\eta'_i} = \Gamma_{\eta_i},\ \frac{d}{dx}\Gamma_{\eta'_{n+1}} = \Gamma_{\eta_{n+1}},\ \frac{d}{dx}\Gamma_{\xi'_1} = \Gamma_{\xi_1},\ \frac{d}{dx}\Gamma_{\xi'_2} = \Gamma_{\xi_2},$$
$$(i = 1,\cdots,n).$$

It follows from the $(n+2)$-nd of these equations that $\mu$ is a constant.

It should now be observed that a minimizing arc for this problem is surely a "normal" arc. For, from the form of the equations (18), it follows that the values of $\eta_0, \eta_{n+1}, \xi_1$ at $x = x_1$, and the value of $\xi_2$ at $x = x_2$, are entirely arbitrary. By hypothesis there exist $r$ sets of admissible variations $(\xi^\rho, \eta^\rho)$, $(\rho = 1,\cdots,r)$, so that the matrix $\| F_\sigma(\xi^\rho, \eta^\rho) \|$, $(\sigma = 2,\cdots,r)$ has rank $r - 1$. Hence from this and the arbitrariness of the end values of $\eta_0, \eta_{n+1}, \xi_1, \xi_2$, as just described, there must exist $r + 3$ sets of admissible variations $\eta_0, \eta_i, \eta_{n+1}, \xi_1, \xi_2$, such that the matrix

$$\| G_\nu(\xi^\delta, \eta^\delta) \|, \qquad (\nu = 2,\cdots,r+3;\ \delta = 1,\cdots,r+3),$$

has rank $r + 2$, where $G_\nu(\xi,\eta) = 0$ are the equations of variation corresponding to the equation (8).

---

[8] 1.

Let $\bar{L}_\delta$ be the $r+3$ constants associated with $g_\delta$ with $\bar{L}_1 = 1$. Let $\theta$ take on successively the values $\eta_0, \eta_i, \eta_{n+1}, \xi_1, \xi_2$. Then from equations (4), the boundary conditions for the auxiliary minimum problem are seen to be

$$\bar{L}_\delta g_{\delta\theta_1} - 2\Gamma_{\theta'}(x_1) = 0, \qquad \bar{L}_\delta g_{\delta\theta_2} + 2\Gamma_{\theta'}(x_2) = 0.$$

From the three equations in which $\theta_2 = \eta_{n+1,2}$, $\theta_2 = \eta_{02}$, $\theta_1 = \eta_{n+1,1}$, it follows that $\bar{L}_{r+3} = -\mu$, $\mu_0 = -1$, $\bar{L}_{r+2} = \mu$, respectively. It then follows from the equation in which $\theta_1 = \eta_{01}$ that $\bar{L}_2 = -1$. When $\theta_2 = \xi_{12}$ and $\theta_1 = \xi_{21}$, it is seen that $\mu_{m+1} = \mu_{m+2} = 0$. Now re-name $\bar{L}_3, \cdots, \bar{L}_{r+1}$, respectively, $L_2, \cdots, L_r$. The remaining boundary conditions have the form

$$
\begin{aligned}
&Q_{\xi_{11}} + L_\sigma(f_{\sigma x_1} + f_{\sigma y_{i1}} y'_{i1}) - 2\mu\xi_{11} = 0, \\
&Q_{\eta_{i1}} + L_\sigma f_{\sigma y_{i1}} \qquad\qquad - 2\Gamma_{\eta'_i}(x_1) = 0, \\
(20)\qquad &Q_{\xi_{22}} + L_\sigma(f_{\sigma x_2} + f_{\sigma y_{i2}} y'_{i2}) - 2\mu\xi_{22} = 0, \\
&Q_{\eta_{i2}} + L_\sigma f_{\sigma y_{i2}} \qquad\qquad + 2\Gamma_{\eta'_i}(x_2) = 0, \\
&\qquad (\sigma = 2, \cdots, r;\ i = 1, \cdots, n).
\end{aligned}
$$

The Euler differential equations of the auxiliary problem are now seen to be

$$(21) \qquad \frac{d}{dx}\Gamma_{\eta'_i} = \Gamma_{\eta_i},$$

where

$$(22) \qquad \Gamma_{\eta'_i} = \omega_{\eta'_i} + \mu_\alpha \phi_{\alpha y'_i}, \qquad \Gamma_{\eta_i} = \omega_{\eta_i} + \mu_\alpha \phi_{\alpha y_i} - \mu\eta_i.$$

It is to be noted that the undetermined constants and multipliers are now $L_\sigma, \mu, \mu_\alpha(x)$ where $(\sigma = 2, \cdots, r;\ \alpha = 1, \cdots, m)$.

The results of this section may be summarized as follows:

*Let $\xi_1, \xi_2, \eta_i(x)$ be a set of functions of class $C'''$ on $x_1 \leqq x \leqq x_2$ which minimizes the second variation $I_2(\xi, \eta)$ in the class of such sets satisfying the differential equations and the end conditions* (16) *and* (17). *Then there must exist m multipliers $\mu_\alpha(x)$, $(\alpha = 1, \cdots, m)$, of class $C'$ and not all identically zero on $x_1x_2$, and r constants $\mu$, $L_\sigma$, $(\sigma = 2, \cdots, r)$, not all zero, satisfying the differential equations* (21) *and the boundary conditions* (20), *in which $\xi_{11}$ and $\xi_{22}$ may be replaced by $\xi_1$ and $\xi_2$ respectively.*

**4. The boundary value problem of the second variation and a necessary condition for the original problem.** From the results of the last section it may be seen that there is a boundary value problem associated with the second variation which may be stated as follows:

To determine multipliers $\mu_\alpha(x)$ of class $C'$ and constants $L_\sigma, \mu$, together with variations $\xi_1, \xi_2, \eta_i(x)$ of class $C'''$ on $x_1x_2$, which satisfy the differential equations

$$(23)\qquad \frac{d}{dx}\Gamma_{\eta'_i} = \Gamma_{\eta_i}, \qquad \Phi_\alpha(x, \eta, \eta') = 0, \frac{d\xi_\iota}{dx} = 0,$$
$$(i = 1, \cdot\cdot\cdot, n;\ \alpha = 1, \cdot\cdot\cdot, m;\ \iota = 1, 2),$$

and the boundary conditions (20) and (8).

It will now be shown that there are restrictions upon the possible values of $\mu$ for which the boundary value problem has solutions, in view of the minimizing properties of the arc $E$ for the original minimum problem considered. Suppose that the set $\xi_\iota$, $\eta_i(x)$, $L_\sigma$, $\mu$ is a non-trivial solution of the boundary value problem and that it also satisfies the equation (17); this last condition could always be met for any given set by multiplying by a suitably chosen positive constant. Multiply the equations (20) by $\xi_{11}$, $\eta_i(x_1)$, $\xi_{22}$, $\eta_i(x_2)$, respectively, and add. Then since $Q$ is a quadratic form in $\xi_{11}$, $\xi_{22}$, $\eta_i(x_1)$ and $\eta_i(x_2)$, and because of the equations (8), there will result

$$(24)\qquad 2Q + 2\eta_i\Gamma_{\eta'_i}\Big|_{x_1}^{x_2} - 2\mu(\xi_{11}{}^2 + \xi_{22}{}^2) = 0.$$

But from equations (22), (15) and (7), it follows that

$$\eta_i\Gamma_{\eta_i} + \eta'_i\Gamma_{\eta'_i} = \eta_i\left(\Gamma_{\eta_i} - \frac{d}{dx}\Gamma_{\eta'_i}\right) + \frac{d}{dx}(\eta_i\Gamma_{\eta'_i})$$
$$= 2\omega + \mu_\alpha\Phi_\alpha - \mu\eta_i\eta_i,$$

whence, in view of the equations (21),

$$\eta_i\Gamma_{\eta'_i}\Big|_{x_1}^{x_2} = \int_{x_1}^{x_2} 2\omega dx - \mu\int_{x_1}^{x_2}\eta_i\eta_i dx.$$

After substituting the right-hand side of this equation in the second term of (24) and recalling that

$$I_2 = \int_{x_1}^{x_2} 2\omega dx + Q,$$

it is seen that

$$2I_2 - 2\mu\left(\xi_{11}{}^2 + \xi_{22}{}^2 + \int_{x_1}^{x_2}\eta_i\eta_i dx\right) = 0,$$

and hence on account of (17) that

$$I_2 = \mu.$$

The following necessary condition on $E$ has thus been proved:

*If $E$ is a minimizing arc for the original minimum problem in xy-space, then the boundary value problem of the second variation described in this section, the equations of which are* (23), (20), *and* (8), *can have no solutions for negative values of the parameter $\mu$.*

**5. Transformation of the boundary value problem of the last section.** The boundary value problem of the last section will now be transformed into a problem that has been discussed by Bliss.[9] It will first be assumed that a solution

$$\xi_\iota,\ \eta_i(x),\ \mu_\alpha(x),\ L_\sigma,\ \mu, \qquad (\iota = 1, 2;\ \alpha = 1, \cdots, m;\ i = 1, \cdots, n)$$

of the boundary value problem of the last section for a minimizing arc $E$ for the original minimum problem has been found. Note that

$$\begin{gathered} F_{\sigma\xi_{11}} = f_{\sigma x_1} + f_{\sigma y_{i1}} y'_{i1}, \quad F_{\sigma\xi_{22}} = f_{\sigma x_2} + f_{\sigma y_{i2}} y'_{i2}, \\ F_{\sigma\eta_{i1}} = f_{\sigma y_{i1}}, \quad F_{\sigma\eta_{i2}} = f_{\sigma y_{i2}}, \qquad (\sigma = 2, \cdots, r). \end{gathered}$$

For convenience, let $\xi_1$, $\xi_2$ be re-named $\eta_{n+1}$, $\eta_{n+2}$, respectively, and adjoin to the boundary value problem of the last section the equations

$$F_{r+1} \equiv \eta_{n+1}(x_1) - \eta_{n+1}(x_2) = 0, \qquad F_{r+2} \equiv \eta_{n+2}(x_1) - \eta_{n+2}(x_2) = 0.$$

It is then clear in view of these equations and hypothesis 4 of section 1 that the matrix

$$\| F_{\tau\eta_{t1}} \ F_{\tau\eta_{t2}} \|, \qquad (\tau = 2, \cdots, r + 2;\ t = 1, \cdots, n + 2), \tag{25}$$

has rank $r + 1$. As a matter of notation, let $\Gamma_{\eta'_i}$ be renamed $\zeta_i$, and for convenience in subsequent proofs, let us introduce two new functions $\zeta_{n+1}$, $\zeta_{n+2}$ defined by the equations

$$\zeta'_{n+1} = \zeta'_{n+2} = 0, \qquad \zeta_{n+1}(x_1) = \zeta_{n+2}(x_1) = 1.$$

The boundary conditions of an equivalent boundary value problem are then

$$\begin{aligned} &L_\sigma F_{\sigma\eta_{i1}} &&+ Q_{\eta_{i1}} &&- 2\zeta_{i1} &&= 0, \\ &L_\sigma F_{\sigma\eta_{n+1,1}} + 2F_{r+1,\eta_{n+1,1}} &&+ Q_{\eta_{n+1,1}} - 2\mu\eta_{n+1,1} &&- 2\zeta_{n+1,1} &&= 0, \\ &2F_{r+2,\eta_{n+2,1}} &&+ Q_{\eta_{n+2,1}} &&- 2\zeta_{n+2,1} &&= 0, \\ &L_\sigma F_{\sigma\eta_{i2}} &&+ Q_{\eta_{i2}} &&+ 2\zeta_{i2} &&= 0, \\ &2F_{r+1,\eta_{n+1,2}} &&+ Q_{\eta_{n+1,2}} &&+ 2\zeta_{n+1,2} &&= 0, \\ &L_\sigma F_{\sigma\eta_{n+2,2}} + 2F_{r+2,\eta_{n+2,2}} &&+ Q_{\eta_{n+2,2}} - 2\mu\eta_{n+2,2} &&+ 2\zeta_{n+2,2} &&= 0, \\ & && F_\tau = 0, && && \\ &(\tau = 2, \cdots, r + 2;\ \sigma = 2, \cdots, r;\ i = 1, \cdots, n). \end{aligned} \tag{26}$$

It is to be observed that $F_\sigma$ and $Q$ do not contain $\eta_{n+1,2}$ and $\eta_{n+2,1}$.

Suppose now that $a_t$ and $b_t$, $(t = 1, \cdots, n + 2)$, are $2n + 4$ constants satisfying the equations

---

[9] 3.

$$(27) \qquad a_t F_{\tau\eta_{t1}} + b_t F_{\tau\eta_{t2}} = 0, \qquad (\tau = 2, \cdots, r+2).$$

In view of the rank of the matrix (25), there will be

$$2n + 4 - (r+1) = 2n + 3 - r$$

linearly independent sets of such constants, say

$$a_{\lambda t}, \; b_{\lambda t}, \qquad (\lambda = 1, 2, \cdots, 2n+3-r).$$

Multiply the first $n+2$ equations of (26) by $a_{\lambda t}$, the second set of $n+2$ equations by $b_{\lambda t}$, and add. The result will be because of (27)

$$\begin{gathered} a_{\lambda s} Q_{\eta_{s1}} - 2a_{\lambda s}\zeta_{s1} - 2a_{\lambda, n+1}\mu\eta_{n+1,1} \\ + b_{\lambda s} Q_{\eta_{s2}} + 2b_{\lambda s}\zeta_{s2} - 2b_{\lambda, n+2}\mu\eta_{n+2,2} = 0, \\ (\lambda = 1, \cdots, 2n+3-r;\; s = 1, \cdots, n+2). \end{gathered}$$

The quadratic form $Q$ with arguments $\eta_{s1}$, $\eta_{s2}$ may be written

$$\begin{gathered} Q(\eta(x_1), \eta(x_2)) = A_{st}\eta_{s1}\eta_{t1} + 2B_{st}\eta_{s1}\eta_{t2} + C_{st}\eta_{s2}\eta_{t2}, \\ (s, t = 1, \cdots, n+2), \end{gathered}$$

where $A_{st}$ and $C_{st}$ may without loss of generality be taken symmetric. It is to be emphasized that $Q$ does not contain $\eta_{n+1,2}$, $\eta_{n+2,1}$. Now define $\| \delta'_{st} \|$ to be the matrix such that $\delta'_{st} = 1$ if $s = t = n+1$ and otherwise zero, and $\| \delta''_{st} \|$ to be the matrix such that $\delta''_{st} = 1$ if $s = t = n+2$ and otherwise zero. The boundary conditions (26) then imply the $2n+4$ conditions

$$\begin{gathered} -a_{\lambda s}\zeta_{s1} + b_{\lambda s}\zeta_{s2} + [a_{\lambda s}(A_{st} - \mu\delta'_{st}) + b_{\lambda s}B_{ts}]\eta_{t1} \\ + [b_{\lambda s}(C_{st} - \mu\delta''_{st}) + a_{\lambda s}B_{st}]\eta_{t2} = 0, \\ F_\tau(\eta) \equiv F_{\tau\eta_{s1}}\eta_{s1} + F_{\tau\eta_{s2}}\eta_{s2} = 0, \\ (\lambda = 1, \cdots, 2n+3-r;\; s, t = 1, \cdots, n+2;\; \tau = 2, \cdots, r+2). \end{gathered}$$

In matrix notation these equations may be written

$$(28) \qquad \begin{gathered} \left\| \begin{matrix} a_{\lambda s}(A_{st} - \mu\delta'_{st}) + b_{\lambda s}B_{ts}, & -a_{\lambda v} \\ F_{\tau\eta_{t1}}, & 0 \end{matrix} \right\| (\eta_{t1}, \zeta_{v1}) \\ + \left\| \begin{matrix} b_{\lambda s}(C_{st} - \mu\delta''_{st}) + a_{\lambda s}B_{st}, & b_{\lambda v} \\ F_{\tau\eta_{t2}}, & 0 \end{matrix} \right\| (\eta_{t2}, \zeta_{v2}) = 0, \\ (s, t, v = 1, \cdots, n+2;\; \lambda = 1, \cdots, 2n+3-r;\; \tau = 2, \cdots, r+2). \end{gathered}$$

The differential equations

$$(29) \qquad \begin{gathered} \frac{d}{dx}\Gamma_{\eta'_i} = \Gamma_{\eta_i}, \qquad \zeta'_{n+1} = \zeta'_{n+2} = 0, \\ \Phi_\alpha(x, \eta, \eta') \equiv \phi_{\alpha y_j}\eta_j + \phi_{\alpha y'_j}\eta'_j = 0, \qquad \eta'_{n+1} = \eta'_{n+2} = 0, \\ (i, j = 1, \cdots, n;\; \alpha = 1, \cdots, m), \end{gathered}$$

will now be reduced to an equivalent set which is for our purposes more useful.[10] Recalling (22) and (15), let $H_{aj}$, $K_{aj}$ be defined with $\zeta_i$ by the equations

$$\zeta_i \equiv \Gamma_{\eta'_i}, \qquad H_{aj} \equiv \phi_{a y_j}, \qquad K_{aj} \equiv \phi_{a y'_j},$$

and assume as usual for regular problems in the calculus of variations that the determinant

$$D = \begin{vmatrix} R_{ij} & K_{\beta i} \\ K_{aj} & L_{a\beta} \end{vmatrix}, \qquad (\alpha, \beta = 1, \cdots, m;\ L_{a\beta} = 0),$$

is different from zero on $x_1 \leqq x \leqq x_2$. Then the $m + n$ equations

$$\begin{aligned} \zeta_i &= Q_{ji}\eta_j + R_{ij}\eta'_j + K_{\beta i}\mu_\beta, \\ 0 &= H_{aj}\eta_j + K_{aj}\eta'_j, \end{aligned}$$

can be solved for $\eta'_j$, $\mu_\beta$. These solutions are

$$\begin{aligned} \eta'_i &= -\frac{1}{D}(R_k{}^i Q_{jk}\eta_j + K_a{}^i H_{aj}\eta_j) + \frac{1}{D} R_j{}^i \zeta_j, \\ \mu_a &= -\frac{1}{D}(K_a{}^k Q_{jk}\eta_j + L_\beta{}^a H_{\beta j}\eta_j) + \frac{1}{D} K_a{}^j \zeta_j, \\ &\qquad (k = 1, \cdots, n), \end{aligned}$$

in which, with reference to $D$, $R_k{}^i$ is the cofactor of $R_{ki}$; $K_a{}^i$, $L_\beta{}^a$, the cofactors, respectively, of $K_{ai}$, $L_{\beta a}$. Note that the matrix $\| R_k{}^i \|$ is symmetric. With the help of (29) it is seen that

$$\zeta'_i = P_{ij}\eta_j + Q_{ij}\eta'_j + H_{ai}\mu_a - \mu\eta_i.$$

When the values of $\eta'_j$ and $\mu_a$ given above are substituted in these equations, they become

$$\begin{aligned} \zeta'_i = P_{ij}\eta_j &- \frac{1}{D} Q_{ih}(R_k{}^h Q_{jk} + K_\nu{}^h H_{\nu j})\eta_j + \frac{1}{D} Q_{ih} R_j{}^h \zeta_j \\ &- \frac{1}{D} H_{\nu i}(K_\nu{}^k Q_{jk} + L_\beta{}^\nu H_{\beta j})\eta_j + \frac{1}{D} H_{\nu i} K_\nu{}^j \zeta_j - \mu\eta_i, \\ &(h = 1, \cdots, n;\ \nu = 1, \cdots, m). \end{aligned}$$

If $D$ is different from zero, the differential equations (29) are then equivalent to the following set in matrix notation:

---

[10] 15, p. 590.

$$
\begin{Vmatrix} & -D^{-1}(R_s{}^q Q_{ls} + K_\alpha{}^q H_{\alpha l}), \\ P_{ql} - D^{-1}Q_{qs}(R_u{}^s Q_{lu} + K_\nu{}^s H_{\nu l}) - D^{-1}H_{\nu q}(K_\nu{}^u Q_{lu} + L_\beta{}^\nu H_{\beta l}), & \\ & D^{-1}R_l{}^q \\ & D^{-1}(Q_{qs}R_l{}^s + H_{\nu q}K_\nu{}^l \end{Vmatrix} (\eta_l, \zeta_l)
$$

$$
+ \mu \begin{Vmatrix} 0 & 0 \\ -\bar{\delta}_{ql} & 0 \end{Vmatrix} (\eta_l, \zeta_l) = \frac{d}{dx}(\eta_q, \zeta_q), \tag{30}
$$

$$
(l, q, s, u = 1, \cdots, n + 2; \quad \alpha, \beta, \nu = 1, \cdots, m),
$$

where $\bar{\delta}_{ql}$ is the Kronecker symbol if $q$ and $l$ are $\leqq n$ and otherwise zero. If any one of $l, q, s, u$ in a coefficient in (30) is greater than $n$, that coefficient is zero.

If $D$ is different from zero, the boundary value problem of the last section can then be transformed into the following boundary value problem:

To determine functions $\eta_q = \eta_q(x)$, $\zeta_q = \zeta_q(x)$, $(q = 1, \cdots, n + 2)$, with $\eta$'s of class $C'''$, $\zeta$'s of class $C''$ on $x_1x_2$, and a constant $\mu$ satisfying the differential equations (30) and the boundary conditions (28). It is clear that, under the hypotheses, every solution of the boundary value problem of section 4 whose equations are (23), (20), and (8), for a minimizing arc $E$ of the original problem, is a solution of the transformed problem just stated.

**6. The self-adjoint character of the auxiliary boundary value problem.** The auxiliary boundary value problem of the last section whose equations are (30) and (28) will now be shown to be "self-adjoint" according to the definition of Bliss.[11] In the paper referred to, it is proved that a necessary and sufficient condition for the self-adjointness of a boundary value problem

$$
\begin{gathered}
\frac{dy_i}{dx} = (A_{i\alpha}(x) + \mu B_{i\alpha}(x))y_\alpha(x), \qquad x_1 \leqq x \leqq x_2, \\
M_{i\alpha}y_\alpha(x_1) + N_{i\alpha}y_\alpha(x_2) = 0,
\end{gathered} \tag{31}
$$

is that there exist a transformation $T_{ik}(x)$ such that

$$
\begin{gathered}
T_{i\alpha}A_{\alpha k} + A_{\alpha i}T_{\alpha k} + T'_{ik} \equiv 0, \qquad T_{i\alpha}B_{\alpha k} + B_{\alpha i}T_{\alpha k} \equiv 0, \\
M_{i\alpha}T^{-1}_{\alpha\beta}(x_1)M_{k\beta} = N_{i\alpha}T^{-1}_{\alpha\beta}(x_2)N_{k\beta}, \\
(i, j, k, \alpha, \beta = 1, \cdots, n).
\end{gathered} \tag{32}
$$

The functions $A_{ik}(x), B_{ik}(x), T_{ik}(x), T'_{ik}(x)$ are assumed to be continuous functions of $x$ with $|T_{ik}| \neq 0$ on $x_1x_2$ and $M_{ik}$, $N_{ik}$ are constants with $n$ the rank of the matrix $\| M_{ik}, N_{ik} \|$.

The boundary value problem given by equations (28) and (30) is evidently of the form

[11] 3, p. 569.

$$
(33)\qquad \begin{aligned}
&\frac{d}{dx}(y_i, z_i) = \left( \begin{Vmatrix} A_{ij} & B_{ij} \\ C_{ij} & D_{ij} \end{Vmatrix} + \mu \begin{Vmatrix} 0 & 0 \\ \bar{C}_{ij} & 0 \end{Vmatrix} \right) (y_j, z_j), \\
&\begin{Vmatrix} E_{pj} & F_{pj} \\ G_{lj} & 0 \end{Vmatrix} (y_j(x_1), z_j(x_1)) + \begin{Vmatrix} \bar{E}_{pj} & \bar{F}_{pj} \\ \bar{G}_{lj} & 0 \end{Vmatrix} (y_j(x_2), z_j(x_2)) = 0,
\end{aligned}
$$

where, in view of (28) and (27), we suppose

$$
(34)\qquad F_{pa} G_{la} - \bar{F}_{pa} \bar{G}_{la} = 0,
$$

$$
(i, j, \alpha = 1, \cdots, n;\ p = 1, \cdots, 2n - r;\ l = 1, \cdots, r).
$$

This is clearly of the same type as (31).

It will now be shown that with the transformation $T$, which with its inverse has the form

$$
T = \begin{Vmatrix} 0 & \delta_{ij} \\ -\delta_{ij} & 0 \end{Vmatrix}, \qquad T^{-1} = \begin{Vmatrix} 0 & -\delta_{ij} \\ \delta_{ij} & 0 \end{Vmatrix},
$$

the equations (32) are satisfied for the boundary value problem of the last section. To do this, we consider what properties the matrices $\| A_{ij} \|, \cdots, \| \bar{G}_{ij} \|$ of (33) must have if the equations (32) are to be satisfied. For the first set of equations of (32) to be satisfied, it is necessary and sufficient that

$$
\begin{Vmatrix} 0 & \delta_{i\alpha} \\ -\delta_{i\alpha} & 0 \end{Vmatrix} \cdot \begin{Vmatrix} A_{\alpha j} & B_{\alpha j} \\ C_{\alpha j} & D_{\alpha j} \end{Vmatrix} + \begin{Vmatrix} A_{\alpha i} & C_{\alpha i} \\ B_{\alpha i} & D_{\alpha i} \end{Vmatrix} \cdot \begin{Vmatrix} 0 & \delta_{\alpha j} \\ -\delta_{\alpha j} & 0 \end{Vmatrix} = 0,
$$

or

$$
\begin{Vmatrix} C_{ij} & D_{ij} \\ -A_{ij} & -B_{ij} \end{Vmatrix} + \begin{Vmatrix} -C_{ji} & A_{ji} \\ -D_{ji} & B_{ji} \end{Vmatrix} = 0.
$$

This is possible if and only if

$$
C_{ij} = C_{ji}, \qquad B_{ij} = B_{ji}, \qquad A_{ij} = -D_{ji};
$$

or, in other words, the matrices $\| C_{ij} \|$ and $\| B_{ij} \|$ must be symmetric, and the matrix $\| A_{ij} \|$ must be equal to the negative of the transpose of the matrix $\| D_{ij} \|$. It readily follows that the second set of equations (32) will be satisfied if and only if

$$
\bar{C}_{ij} = \bar{C}_{ji},
$$

that is, the matrix $\| C_{ij} \|$ must be symmetric. These conditions are easily verified to be fulfilled for the corresponding matrices of the equations (30). Hence for the transformation $T$, as defined above, the first two sets of equations (32) are satisfied for the boundary value problem of the last section.

The left member of the third equation of (32) for the problem (33) is

$$\begin{Vmatrix} E_{p\alpha} & F_{p\alpha} \\ G_{l\alpha} & 0 \end{Vmatrix} \cdot \begin{Vmatrix} 0 & -\delta_{\alpha\beta} \\ \delta_{\alpha\beta} & 0 \end{Vmatrix} \cdot \begin{Vmatrix} E_{q\beta} & G_{m\beta} \\ F_{q\beta} & 0 \end{Vmatrix}$$

$$= \begin{Vmatrix} F_{p\beta} & -E_{p\beta} \\ 0 & -G_{l\beta} \end{Vmatrix} \cdot \begin{Vmatrix} E_{q\beta} & G_{m\beta} \\ F_{q\beta} & 0 \end{Vmatrix},$$

$$(p, q = 1, \cdots, 2n - r; \quad l, m = 1, \cdots, r; \quad \alpha, \beta = 1, \cdots, n),$$

and this last product is equal to

$$\begin{Vmatrix} F_{p\beta}E_{q\beta} - E_{p\beta}F_{q\beta}, & F_{p\beta}G_{m\beta} \\ -G_{l\beta}F_{q\beta} & 0 \end{Vmatrix}.$$

Similarly the right member of the third equation of (32) for the problem (33) is found to be

$$\begin{Vmatrix} \bar F_{p\beta}\bar E_{q\beta} - \bar E_{p\beta}\bar F_{q\beta}, & \bar F_{p\beta}\bar G_{m\beta} \\ -G_{l\beta}\bar F_{q\beta}, & 0 \end{Vmatrix}.$$

These two matrices will be equal if and only if

$$(35) \qquad \begin{aligned} F_{p\beta}G_{m\beta} &= \bar F_{p\beta}\bar G_{m\beta}, \\ F_{p\beta}E_{q\beta} - E_{p\beta}F_{q\beta} &= \bar F_{p\beta}\bar E_{q\beta} - \bar E_{p\beta}\bar F_{q\beta}. \end{aligned}$$

The first equation is true because of (34). The second equation may be verified for the boundary value problem of the last section. For, let the range of the subscripts of the last equations be

$$\alpha, \beta = 1, \cdots, n + 2; \quad p, q = 1, \cdots, 2n + 3 - r; \quad l, m = 2, \cdots, r + 2.$$

Then substituting in (35) the values which make the matrices in (33) identical with those in (28), we find that the left and right members, respectively, of (35) are

$$(-a_{p\beta})(a_{qs}(A_{s\beta} - \mu\delta'_{s\beta}) + b_{qs}B_{\beta s}) - (a_{ps}(A_{s\beta} - \mu\delta'_{s\beta}) + b_{ps}B_{\beta s})(-a_{q\beta}),$$
$$(b_{p\beta})(b_{qs}(C_{s\beta} - \mu\delta''_{s\beta}) + a_{qs}B_{s\beta}) - (b_{ps}(C_{s\beta} - \mu\delta''_{s\beta}) + a_{ps}B_{s\beta})(b_{q\beta}).$$

Because of the symmetry of the matrices $\| A_{st} \|$, $\| C_{st} \|$, $\| \delta'_{st} \|$ and $\| \delta''_{st} \|$, these two expressions are equal if

$$a_{p\beta}b_{qs}B_{\beta s} = a_{ps}B_{s\beta}b_{q\beta},$$

and

$$b_{ps}B_{\beta s}a_{q\beta} = b_{p\beta}a_{qs}B_{s\beta}.$$

These, however, are true equations, as we may see by interchanging certain of the summation subscripts, and hence the third equation of (32) is satisfied for the boundary value problem of the last section.

All three of the equations (32) are then satisfied for the auxiliary boundary value problem of the last section, from which it follows that this problem is self-adjoint.

The auxiliary boundary value problem is, moreover, "definitely" self-adjoint, according to the definition of Bliss. Let $S_{ik}$ be defined by $S_{ik}(x) = T_{\alpha i}B_{\alpha k}$. The boundary value problem (31) is then said to be "definitely" self-adjoint[12] if the following conditions are fulfilled: (1) The equations (32) are satisfied for the transformation $T_{ik}(x)$. (2) The matrix $\| S_{ik}(x) \|$ is symmetric. (3) The bilinear form $S_{\alpha\beta}(x) f_\alpha \bar{f}_\beta$ formed for a set of numbers $f_i$ and their conjugate imaginaries $\bar{f}_i$ is positive or zero at every point of $x_1x_2$. (4) This form vanishes identically for a set of solutions $f_i(x)$ of a system of equations

$$f'_i(x) = A_{i\alpha}(x) f_\alpha(x) + B_{i\alpha}(x) g_\alpha(x), \qquad (i, k, \alpha, \beta = 1, \cdots, n),$$

where the $g_i$'s are continuous functions of $x$ on $x_1x_2$ but otherwise arbitrary, only when the functions $f_i(x)$ are all identically zero. These conditions are easily seen to be satisfied for the auxiliary boundary value problem of the last section since $S$ has the value

$$S = \begin{Vmatrix} 0 & -\delta_{qv} \\ \delta_{qv} & 0 \end{Vmatrix} \cdot \begin{Vmatrix} 0 & 0 \\ -\bar{\delta}_{vl} & 0 \end{Vmatrix} = \begin{Vmatrix} \bar{\delta}_{ql} & 0 \\ 0 & 0 \end{Vmatrix}, \qquad (q, v, l = 1, \cdots, n+2).$$

The theory of definitely self-adjoint boundary value problems as developed by Bliss[13] may now be applied in full to the problem of the last section. According to that theory, the characteristic constants $\mu$ for which this problem has solutions must all be real and are denumerably infinite in number, and the linearly independent characteristic solutions corresponding to each characteristic constant may be chosen real. Since every solution of the problem of section 4 is also a solution of the problem of the last section, it follows that the characteristic constants $\mu$ for that problem are also all real, and the linearly independent characteristic solutions corresponding to each characteristic constant may be chosen real. By the criterion of section 4, none of the characteristic parameter values for the problem of section 4 can be negative if $E$ is a minimizing arc for the original problem in $xy$-space.

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[12] 3, p. 570.

[13] 3, p. 571.

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# ON THE ASYMPTOTIC DISTRIBUTION OF $\zeta'/\zeta(s)$ IN THE CRITICAL STRIP.*

By RICHARD KERSHNER and AUREL WINTNER.

According to Bohr,[1] Euler's product for $\zeta(s)$, which is divergent for $\sigma < 1$, has for $\sigma > \frac{1}{2}$ certain convergence tendencies. In particular, the formal expansion of $\log \zeta(s)$ is, for every fixed $\sigma > \frac{1}{2}$, convergent in relative measure to $\log \zeta(s)$, and this holds uniformly for all $\sigma >$ const. $> \frac{1}{2}$. Bohr's proof of this fact consists in first replacing $\zeta(s)$ by

$$(1 - 2^{1-s})\zeta(s) = \sum_{n=1}^{\infty} (-1)^{n+1}/n^s, \text{ where } \sigma > 0,$$

then applying Schnee's mean-value theorem uniformly for $\sigma >$ const. $> \frac{1}{2}$, and finally removing the factor $(1 - 2^{1-s})$. Thus the method does not apply to the function[2] $\zeta'/\zeta(s)$, a function more directly connected with the prime number distribution than $\zeta(s)$ itself.

In the present note there will first be obtained results for $\zeta'/\zeta(s)$ which are analogous to those of Bohr for $\log \zeta(s)$. This will be made possible in view of a general principle which may be formulated roughly as follows: One cannot lose uniform convergence in relative measure by term-by-term differentiation of a sequence of analytic functions (the same does not hold for term by term integration).

The result is then applied to obtain for the asymptotic distribution of $\zeta'/\zeta(s)$, where $\sigma > \frac{1}{2}$, results which are analogous to those previously obtained[3] for $\log \zeta(s)$ or $\zeta(s)$. Due to the convergence in relative measure, these results follow immediately from the general theory of infinite convolutions,[3] since the logarithms of the prime numbers are linearly independent. The asymptotic distribution of $\zeta'/\zeta(s)$ has recently[4] been discussed for $\sigma > 1$. The facts to be proved seem to be new not only for $\frac{1}{2} < \sigma < 1$ but for $\sigma = 1$ as well.

The results are independent of Riemann's hypothesis.

Let $s_n = \sigma_n + it_n$ denote those zeros, if any, of $\zeta(s)$ for which $\sigma_n > \frac{1}{2}$, and let $J_n$ be the interval

$$t = t_n, \qquad \tfrac{1}{2} < \sigma \leqq \sigma_n \; (< 1),$$

* Received January 29, 1937.

[1] Bohr [1], [2].

[2] By $\zeta'/\zeta(s)$ is meant $\zeta'(s)/\zeta(s)$.

[3] Jessen and Wintner [3].

[4] van Kampen and Wintner [4], Section 6.

an interval which is perpendicular to the critical line. Let $\Gamma$ denote the set obtained from the half-plane $\sigma > \frac{1}{2}$ by removing the segments $J_n$ and also the segment

$$t = 0, \qquad \tfrac{1}{2} < \sigma \leqq 1.$$

Thus $\Gamma$ is open, simply-connected, and contains the half-plane $\sigma > 1$. On placing

$$(1) \qquad \zeta_n(s) = \zeta(s) \prod_{k=1}^{n} (1 - p_k^{-s}); \qquad (n = 0, 1, 2, \cdots)$$

it is clear that

$$(2) \qquad \zeta_n(s) \neq 0 \text{ in } \Gamma \qquad (\zeta_0 = \zeta).$$

Let $\log \zeta_n(s)$ denote that logarithm of $\zeta_n(s)$ which is regular analytic in $\Gamma$ and vanishes as $\sigma \to +\infty$. The considerations of the sequel will be based on a classical result concerning the manner of divergence of Euler's product in the critical strip, namely on

*Bohr's Lemma.*[5] Corresponding to three arbitrary numbers $\eta$, $\epsilon_1$, $\epsilon_2$ satisfying

$$0 < \eta < \tfrac{1}{2}; \quad 0 < \epsilon_1 < 1, \quad 0 < \epsilon_2 < 1,$$

there exists an $N = N(\eta, \epsilon_1, \epsilon_2)$ with the property that one can choose for every $n \geqq N$ a $u_n$ such that if $T \geqq u_n$, then the interval $2 \leqq \tau \leqq T$ contains a finite number of mutually disjoint subintervals $I$ which have a total length greater than $(1 - \epsilon_2)T$ and are such that if $\tau$ is in an $I$, then on the one hand the half-strip

$$\sigma \geqq \tfrac{1}{2} + \eta, \qquad \tau - \tfrac{1}{2} \leqq t \leqq \tau + \tfrac{1}{2}$$

of the $(\sigma + it)$-plane is contained in $\Gamma$ and on the other hand

$$|\log \zeta_n(s)| \leqq \epsilon_1$$

for every $s = \sigma + it$ in any of these half-strips.

Let $\Sigma_0$ denote the boundary of a square of side $\frac{1}{2}$ in the complex $s$-plane and let $\Sigma_\delta$, where $0 < \delta < \frac{1}{4}$, denote the boundary of that square of side $\frac{1}{2} - 2\delta$ which is symmetrically placed in $\Sigma_0$. Then, if $f(s)$ is any function regular analytic on and within $\Sigma_0$, its derivative $f'(s)$ satisfies the inequality

$$(3) \qquad \max_{\Sigma_\delta} |f'(s)| \leqq \frac{1}{\delta^2} \max_{\Sigma_0} |f(s)|, \qquad (0 < \delta < \tfrac{1}{4}),$$

regardless of the position of $\Sigma_0$ in the $s$-plane. In fact, if $s$ is in the interior of $\Sigma_0$, then, according to Cauchy,

[5] Bohr [1], Hilfssatz 5, p. 82. The numbers denoted above by $\epsilon_1, \epsilon_2, \eta, N, u_n$ are in Bohr's notation $\epsilon', \epsilon'', \sigma' - \frac{1}{2}, N^*, T^*$ respectively.

$$2\pi i f'(s) = \int_{\Sigma_0} (s-w)^{-2} f(w)\,dw.$$

Now if $s$ is on or within $\Sigma_\delta$, then $|s-w| \geqq \delta$ for every $w$ on $\Sigma_0$, so that (3) is obvious. Let $\Sigma_0$ be any square of side $\frac{1}{2}$ which has one side on the line $\sigma = \frac{1}{2} + \eta$ and is contained in one of the half-strips mentioned in Bohr's Lemma. On applying (3) to each of these squares $\Sigma_0$ and to the function $f(s) = \log \zeta_n(s)$, it is seen that Bohr's Lemma implies the following:

Corresponding to four arbitrary numbers $\delta$, $\eta$, $\epsilon_1$, $\epsilon_2$ satisfying

$$0 < \delta < \tfrac{1}{4}, \quad 0 < \eta < \tfrac{1}{2}, \quad 0 < \epsilon_1 < 1, \quad 0 < \epsilon_2 < 1,$$

there exists an $N = N(\eta, \epsilon_1, \epsilon_2)$ with the property that one can choose for every $n \geqq N$ a $u_n$ such that if $T \geqq u_n$, then the interval $2 \leqq \tau \leqq T$ contains a finite number of mutually disjoint subintervals $I$ which have a total length greater than $(1-\epsilon_2)T$ and are such that if $\tau$ is in an $I$, then on the one hand the square

$$\tfrac{1}{2} + \eta + \delta \leqq \sigma \leqq 1 + \eta - \delta, \qquad \tau - \tfrac{1}{4} + \delta \leqq t \leqq \tau + \tfrac{1}{4} - \delta$$

is contained in $\Gamma$ and on the other hand

$$|\zeta'_n/\zeta_n(s)| \leqq \epsilon_1/\delta^2$$

for every $s = \sigma + it$ in any of these squares.

Now if $\sigma_0$ is a fixed value such that $\frac{1}{2} < \sigma_0 \leqq 1$, one can choose $\eta > 0$ and $\delta > 0$ such that

$$\tfrac{1}{2} + \eta + \delta < \sigma_0 < 1 + \eta - \delta, \qquad (\eta < \tfrac{1}{2},\ \delta < \tfrac{1}{4}).$$

Let $\delta$ and $\eta$ be fixed. Then, on placing $\epsilon^1 = \epsilon_1/\delta^2$ and applying the above result only for a fixed $\sigma = \sigma_0$, one obtains the following corollary: If $\sigma_0 > \frac{1}{2}$ is fixed,[6] there exists for every pair of positive numbers $\epsilon^1$, $\epsilon_2$ an $N = N(\epsilon^1, \epsilon_2)$ with the property that one can choose for every $n \geqq N$ a $u_n$ such that if $T \geqq u_n$, then

$$|\zeta'_n/\zeta_n(\sigma_0 + it)| \leqq \epsilon^1$$

whenever $t$ is on a certain subset of the interval $2 \leqq t \leqq T$, and this subset has a measure greater than $(1-\epsilon_2)T$. Since $\epsilon^1, \epsilon_2$ are independent of each other, it follows, by letting $\epsilon_2 \to 0$, $T \to +\infty$ and writing $\sigma$ and $\epsilon$ instead of $\sigma_0$ and $\epsilon^1$, that, if $\sigma > \frac{1}{2}$ is fixed, then, for every fixed $\epsilon > 0$,

$$(4) \qquad \lim_{n\to+\infty} \limsup_{T\to+\infty} \frac{1}{T} \operatorname*{meas}_{0<t<T} \{|\zeta'_n/\zeta_n(\sigma+it)| > \epsilon\} = 0,$$

where the factor of $1/T$ denotes the linear measure of the set of those values

[6] If $\sigma_0 > 1$, the statement is trivial.

$t$ for which

$$0 < t < T \quad \text{and} \quad |\zeta'_n/\zeta_n(\sigma + it)| > \epsilon.$$

Thus (4) means that if $\sigma > \frac{1}{2}$ is fixed, the sequence

$$\zeta'_1/\zeta_1(\sigma + it), \cdots, \zeta'_n/\zeta_n(\sigma + it), \cdots$$

tends in relative measure [7] to the function of $t$ which is $\equiv 0$. It is clear from the above proof that the convergence in relative measure is uniform for all $\sigma \geqq \bar{\sigma}$, if $\bar{\sigma} > \frac{1}{2}$ is fixed. On writing

$$f_n(t)[\rightarrow]f(t), \qquad n \rightarrow +\infty,$$

if $f_n(t)$ tends in relative measure to $f(t)$, i. e., if

$$\lim_{n\rightarrow+\infty} \limsup_{T\rightarrow+\infty} \frac{1}{T} \operatorname*{meas}_{0<t<T} \{|f(t) - f_n(t)| > \epsilon\} = 0$$

holds for every fixed $\epsilon > 0$, the above result may be formulated as follows:

THEOREM I. *If $\sigma > \frac{1}{2}$ is fixed, then Euler's series*

$$(5) \qquad -\sum_{k=1}^{\infty} \frac{p_k^{-s} \log p_k}{1 - p_k^{-s}}, \qquad (s = \sigma + it),$$

*for the function $\zeta'/\zeta(s)$ converges in relative measure to $\zeta'/\zeta(\sigma + it)$,* i. e.,

$$(6) \qquad \rho_n(\sigma + it)[\rightarrow]\zeta'/\zeta(\sigma + it), \qquad n \rightarrow +\infty,$$

*where $\rho_n$ is an abbreviation for*

$$(7) \qquad \rho_n(s) = \rho_n(\sigma + it) = -\sum_{k=1}^{n} \frac{p_k^{-s} \log p_k}{1 - p_k^{-s}}.$$

*Furthermore,* (4) *holds uniformly for $\sigma \geqq \bar{\sigma}$, if $\bar{\sigma} > \frac{1}{2}$ is fixed.*

In fact, if $\sigma > 1$, then (5) may be written as the absolutely convergent Dirichlet series of $\zeta'/\zeta(s)$, and so

$$(8) \qquad \zeta'_n/\zeta_n(s) = \zeta'/\zeta(s) - \rho_n(s)$$

is, for $\sigma > 1$, obvious from the definitions (6) and (1) of $\rho_n(s)$ and $\zeta_n(s)$. Now the function (7) is regular analytic in the half-plane $\sigma > 0$, and this half-plane contains the domain $\Gamma$ defined above. Since $\zeta(s)$ and $\zeta_n(s)$ are regular analytic and distinct from zero in the simply connected domain $\Gamma$ which contains the half-plane $\sigma > 1$, it follows that (8) holds at every point $s$ of $\Gamma$. Consequently, (6) is equivalent to (4).

---

[7] Cf. Jessen and Wintner [3], Section 11.

It may be mentioned that the function $\zeta'/\zeta(\sigma+it)$ to which the series (5) is convergent in relative measure is a continuous function of $t$, if $\sigma > \frac{1}{2}$ is not the abscissa of a zero of $\zeta$ (supposing that there exist such zeros). The corresponding fact does not hold in Bohr's case of $\log \zeta(s)$, if the Riemann hypothesis is false. In fact, if the Riemann hypothesis is false, then each point of the cuts $J_n$ occurring in the definition of $\Gamma$ is a discontinuity of the function $\log \zeta(\sigma+it)$ of $t$, if $\sigma > \frac{1}{2}$ is fixed and $\sigma < \sigma_n$, where $\sigma_n$ is the abscissa of the right-hand end of $J_n$.

*Remark.* Since $\zeta'/\zeta(s)$ is regular for $s = 1 + it \neq 1$, and since the series

$$\sum_{n=1}^{\infty} \{(1 - p_n^{-s})^{-1} p_n^{-s} \log p_n - p_n^{-s} \log p_n\} = \sum_{n=1}^{\infty} \sum_{k=2}^{\infty} p_n^{-ks} \log p_n$$

represents a regular function for $\sigma > \frac{1}{2}$ and so for $s = 1 + it$, it is clear from

$$\sum_{n=1}^{m} p_n^{-1} \log p_n \sim \log p_m = O(\log m), \qquad m \to +\infty,$$

that the series

$$-\sum_{n=1}^{\infty} p_n^{-s} \log p_n = \zeta'/\zeta(s) + \sum_{n=1}^{\infty} \sum_{k=2}^{\infty} p_n^{-ks} \log p_n$$

is, in virtue of a general theorem of M. Riesz,[8] convergent for every $s = 1 + it \neq 1$. In other words, the series (5) is convergent for every $s = 1 + it \neq 1$. This does not imply, however, the statement of Theorem I for $\sigma = 1$, since (5) is not uniformly convergent for all $s = 1 + it$, $2 < t < +\infty$ (if $\sigma < 1$, then (5) is clearly divergent for every $t$). Correspondingly, the following theorem seems to be new not only for $\frac{1}{2} < \sigma < 1$ but for $\sigma = 1$ as well:

THEOREM II. *The function $\zeta'/\zeta(\sigma+it)$ possesses, for every fixed $\sigma > \frac{1}{2}$, an asymptotic distribution function.*

This theorem is[9] a consequence[10] of Theorem I, since the function $\rho_n(\sigma+it)$ of $t$ is, according to (7), almost periodic in the sense of Bohr.

THEOREM III. *The asymptotic distribution function of $\zeta'/\zeta(\sigma+it)$, where $\sigma > \frac{1}{2}$ is fixed, is absolutely continuous with a density which possesses partial derivatives of arbitrarily high order. If $\frac{1}{2} < \sigma < 1$, then the density is a transcendental entire function of two variables.*

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[8] M. Riesz [5], p. 350.

[9] Cf. Jessen and Wintner [3], Section 11.

[10] The statement of Theorem I as to uniformity with respect to $\sigma$ is not needed.

In fact, since the logarithms of the prime numbers are linearly independent, Theorem III follows from Theorem I by an obvious modification of methods previously applied [11] to $\log \zeta(\sigma + it)$. The same holds [11] also for

THEOREM IV. *If* $\frac{1}{2} < \sigma \leqq 1$, *the density of the asymptotic distribution function of the function* $\zeta'/\zeta(\sigma + it)$ *of* $t$ *is everywhere positive but vanishes in the infinity as strongly as a Gaussian density.*

Needless to say, Theorem IV may be considered as an indication of the truth of Riemann's hypothesis (which, in turn, does not imply Theorem IV).

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[11] Jessen and Wintner [3], Sections 8, 13, 14.

# ON THE ADDITION OF CONVEX CURVES AND THE DENSITIES OF CERTAIN INFINITE CONVOLUTIONS.*

By E. R. van Kampen.

Let $S_1, S_2, \cdots$ be a sequence of convex curves in a plane such that the infinite convolution $\phi$ of the distribution functions[1] $\phi_1, \phi_2, \cdots$ corresponding to $S_1, S_2, \cdots$ is convergent. Under very general conditions it has been proved[2] that the distribution function $\phi$ has continuous partial derivatives of arbitrarily high order. If the curves $S_1, S_2, \cdots$ are analytic, the question arises whether $\phi$ is regular analytic anywhere in the plane. In certain special cases this question has already been investigated.[3] The object of the present paper is to develop a concise formalism for the description of the geometrical process of addition of convex curves. On using this formalism in connection with the study of $\phi$ a simple proof will be given in Section IV for the analyticity of the density of $\phi$ in certain specified regions. Although the boundaries of these regions are defined in such a way as to suggest their having a singular character for the density of $\phi$, it still remains a problem to decide under fairly general conditions whether or not it is possible for $\phi$ to be regular on these boundaries.[4] As an application it will be shown in Section V that there exist for the asymptotic distribution function of the logarithm of the Riemann zeta function in addition to the ring shaped regions of analyticity previously determined,[5] certain regions of analyticity which may be described as crescent shaped.

In Sections I, II, III, in which the geometric considerations are given, the convex curves have been replaced by convex hypersurfaces in an $n$-dimensional space. This modification has no effect on the proofs but justifies the introduction of vector notations which lead to a slight simplification even if $n = 2$. The content of Section I is in the main a repetition of a treatment of the same problem by Kershner.[6] However, apart from the minor differences concerning the dimension of the containing space and the notation, the complete induction process used by Kershner has been replaced by a direct passage from the case

* Received March 15, 1937.

[1] Cf. Jessen and Wintner [3], Sections 4 and 7.

[2] *Ibid.*, Section 8.

[3] Van Kampen and Wintner [6].

[4] *Ibid.*, Appendix.

[5] *Ibid.*, Section 7.

[6] Kershner [4].

of 2 hypersurfaces to the general case, so that a limiting process for the case of infinitely many hypersurfaces becomes superfluous. The result of Section III is not used in what follows, but gives an insight in the character of the sets $F$ and $G$ introduced earlier.

**I. Vectorial sums of convex hypersurfaces.** Let $\omega$, $\eta$ be variable vectors in a real finite-dimensional vector space such that $\omega$ is of length 1 and let $\omega \cdot \eta$ denote the scalar product of these two vectors. By a convex hypersurface will be understood the point set theoretic boundary of a non-empty bounded convex point set in the space. Thus in particular a convex hypersurface may degenerate into a single point.

Let $S_k$ be a convex hypersurface. If the exterior normal to a supporting hyperplane of $S_k$ at the point $\eta$ of $S_k$ has the direction determined by $\omega$, the notation

$$\eta = \eta_k(\omega) \tag{1}$$

will be used, even though (1) is in general not an allowable parametric representation of $S_k$. The supporting function $h_k(\omega)$ of $S_k$ appears in the form

$$h_k(\omega) = \omega \cdot \eta_k(\omega). \tag{2}$$

The convex hypersurface $S^*_k$ which is symmetrical to $S_k$ with respect to the origin may be represented by

$$\eta = -\eta_k(-\omega), \tag{3}$$

and so its supporting function is

$$h^*_k(\omega) = h_k(-\omega). \tag{4}$$

The fact that $S_k$ is convex may be expressed by the inequality

$$\omega \cdot \eta_k(\omega') \leqq h_k(\omega), \tag{5}$$

where the equality sign holds for $\omega = \omega'$. On substituting $-\omega$ in (5) instead of $\omega$ one obtains

$$-h_k(-\omega) \leqq \omega \cdot \eta_k(\omega') \tag{6}$$

where the equality sign holds for $\omega = -\omega'$, in view of (2). It follows from (5) and (6) that

$$h_k(\omega) + h_k(-\omega) \geqq 0, \tag{7}$$

which relation is also evident from the convexity of $S_k$. A point $\eta$ clearly is contained in the closed convex set determined by $S_k$ if and only if

(8) $$\omega \cdot \eta \leqq h_k(\omega),$$

for every $\omega$, while such a point $\eta$ is a point $\eta = \eta_k(\omega')$ of $S_k$ if and only if the equality sign in (8) holds for $\omega = \omega'$.

Let an infinite sequence of convex hypersurfaces

(9) $$S_1, S_2, \cdots$$

be given by (1) such that

(10) $$|\eta_k(\omega)| < a_k, \qquad \Sigma_k a_k < +\infty,$$

for certain numbers $a_k$ and all $\omega$. Thus all series occurring in this Section are absolutely (uniformly) convergent. The particular case where only a finite number of hypersurfaces are given need not be excluded but will not be referred to. By a simple argument [7] it may be seen that the locus $S$ represented by [8]

(11) $$\eta = \Sigma_k \eta_k(\omega)$$

is a convex hypersurface and has the supporting function

(12) $$h(\omega) = \Sigma_k h_k(\omega).$$

The locus represented by

(13) $$\eta = \Sigma_k \eta_k(\omega_k),$$

where the $\omega_k$ vary independently, is the vectorial sum of the $S_k$ and will be denoted by $T$. It follows from (5) and (13) by summation that

(14) $$\omega \cdot \eta \leqq \Sigma_k h_k(\omega) = h(\omega),$$

for every point $\eta$ in $T$, while in view of (2) the equality sign holds in (14) if $\omega_k = \omega$ for every $k$, in which case $\eta$ is a point in $S$ also. Thus $T$ is contained in the closed convex set $C$ determined by $S$. On adding (2) for $k = 1$ and (6) for all $k > 1$, it follows from (13) that

$$h_1(\omega) - \sum_{k>1} h_k(-\omega) \leqq \omega \cdot \eta$$

if $\eta$ is determined by (13) and $\omega = \omega_1$. Thus the set $U$ of points $\eta$ defined by the inequality

(15) $$\omega \cdot \eta < h_1(\omega) - \sum_{k>1} h_k(-\omega)$$

[7] Haviland [2].

[8] This equation should be read in the sense that if for a given $\omega$ and for one or more values of $k$ the functions $\eta_k(\omega)$ are not univalued, then (11) represents all possible values taken by the sum on the right.

does not have a point in common with $T$ although $U$ is contained in $C$ as a consequence of (7). From the form of (15) it is clear that the set $U$ is either empty or an open convex set. On replacing $\omega$ in (15) by $-\omega$ and adding the result to (15) one obtains

$$(16) \qquad h_1(\omega) + h_1(-\omega) > \sum_{k>1} h_k(\omega) + h_k(-\omega),$$

so that (16) is a necessary (but not sufficient) condition for the set $U$ to be non-empty. It follows also that $U$ must be empty except for at most one choice of $S_1$ among the given $S_k$. In order to assure that the correct choice of $S_1$ has been made, it will be assumed that the $S_k$ have been enumerated in such a way that

$$(17) \quad \mathrm{Max}_\omega\{h_k(\omega) + h_k(-\omega)\} \leqq \mathrm{Max}_\omega\{h_{k+1}(\omega) + h_{k+1}(-\omega)\}, \quad (k = 1, 2, \cdots)$$

A boundary point $\eta_0$ of $U$ clearly satisfies the inequality

$$(18) \qquad \omega \cdot \eta_0 \leqq h_1(\omega) - \sum_{k>1} h_k(-\omega),$$

while the equality sign in (18) holds for at least one value of $\omega$, say for $\omega = \omega_0$. Put

$$(19) \qquad \eta' = \sum_{k>1} \eta_k(-\omega_0) \quad \text{and} \quad \eta_0 = \eta' + \eta''.$$

On placing in (6) the variable $\omega'$ equal to $-\omega_0$ and adding the result to (18) for all $k > 1$, one obtains in view of (19)

$$\omega \cdot \eta_0 - \omega \cdot \eta' = \omega \cdot \eta'' \leqq h_1(\omega),$$

where the equality sign still holds for $\omega = \omega_0$. Thus $\eta' = \eta_1(\omega_0)$, by the remark made in connection with (8), so that $\eta_0$ is a point of the locus $R'$ represented by

$$(20) \qquad \eta = \eta_1(\omega) + \sum_{k>1} \eta_k(-\omega).$$

Since $R'$ is contained in $T$ it is clear that the boundary $R$ of $U$ is contained in $T$. Now it will be shown that $T = C - U$.

Consider first the case where $k = 1, 2$ only and let $\eta_0$ be a point which is in $C$ but not in $T$. Obviously

$$(21) \qquad \eta_0 - \eta_2(-\omega_2) \neq \eta_1(\omega_1),$$

for every $\omega_1, \omega_2$, so that the vectorial sum $T'$ of the convex hypersurfaces represented by the point $\eta_0$ and by the convex hypersurface $S^*_2$ does not have a point in common with $S_1$. Obviously $T'$ is obtained from $S^*_2$ by a trans-

lation along the vector $\eta_0$. It is impossible that $T'$ and $S_1$ are exterior to each other, since otherwise the inequality (21) would hold for all $\eta$ on a line joining $\eta_0$ with $\infty$, hence no point of this line would be in $T$. This contradicts the assumption that $\eta_0$ is in $C$, since the boundary of $C$ is in $T$. Also $S_1$ cannot be interior to $T'$, by (17). Thus $T'$ must be interior to $S_1$. This implies

$$\omega \cdot \eta_0 + h_2(-\omega) < h_1(\omega), \tag{22}$$

so that $\eta_0$ is a point of $U$, and the statements that $T = C - U$ is proved if only two hypersurfaces are considered.

In the general case,[9] let again $\eta_0$ be a point in $C$ but not in $T$, so that

$$\omega \cdot \eta_0 < \Sigma_k h_k(\omega), \tag{23}$$

for all $\omega$, since $\eta_0$ is in $C$ but not on the boundary $S$ of $C$ and

$$\eta_0 \neq \Sigma_k \eta_k(\omega_k), \tag{24}$$

for any sequence $\omega_1, \omega_2, \cdots$. It must be shown that $\eta_0$ satisfies (15). The inequality (24) may be written in the form

$$\eta_0 - \Sigma'' \eta_k(-\omega_k) \neq \Sigma'(\omega_k), \tag{25}$$

where $\Sigma'$ denotes a summation over a set of distinct positive integers $k$ and $\Sigma''$ denotes the summation over the remaining positive integers $k$. It may be seen from (25) (by a repetition of the argument which follows (21)) that either

$$\omega \cdot \eta_0 > \Sigma' h_k(\omega) - \Sigma'' h_k(-\omega) \tag{26a}$$

or

$$\omega \cdot \eta_0 < \Sigma' h_k(\omega) - \Sigma'' h_k(-\omega) \tag{26b}$$

holds for given summations $\Sigma'$, $\Sigma''$ and for every $\omega$.

Now suppose, if possible, that $\eta_0$ is not in $U$, so that $\eta_0$ does not satisfy (15). Then the alternatives (26a), (26b) implies that

$$\omega \cdot \eta_0 > h_1(\omega) - \sum_{k>1} h_k(-\omega). \tag{27}$$

On the other hand it follows from (20) and from the absolute-uniform convergence of the series involved that

$$\omega \cdot \eta_0 < \sum_{k \leqq p} h_k(\omega) - \sum_{k>p} h_k(-\omega), \tag{28}$$

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[9] Only at this point the treatment becomes essentially different from the one given by Kershner [4].

for a sufficiently large $p$ and for every $\omega$. Thus the alternatives (23a), (23b) and (7) imply the existence of an integer $l$ such that

$$(29) \quad \sum_{k<l} h_k(\omega) - h_l(-\omega) < \omega \cdot \eta_0 + \sum_{k<l} h_k(-\omega) < \sum_{k<l} h_k(\omega) + h_l(\omega),$$

for every $\omega$. On denoting by $S^1$, $S^2$ the convex hypersurfaces represented by

$$S^1: \quad \eta^1(\omega) = \sum_{k<l} \eta_k(\omega); \qquad h^1(\omega) = \sum_{k<l} h_k(\omega)$$
$$S^2: \quad \eta^2(\omega) = \eta_l(\omega); \qquad h^2(\omega) = h_l(\omega)$$

it is clear from (17) and (7) that the condition corresponding to (17) in the case of $S^1$, $S^2$ is satisfied, so that the result obtained above in the case of two hypersurfaces may be applied to $S^1$, $S^2$. Thus, by (29) the convex hypersurface represented by

$$\eta = \eta_0 - \sum_{k>l} \eta_k(\omega)$$

is contained in the vectorial sum of $S^1$ and $S^2$. But this is in contradiction with the assumption that $\eta_0$ is not in $T$. This completes the proof of the following

THEOREM 1. *Let $S_k$ $(k = 1, 2, \cdots)$, be a sequence of convex hypersurfaces the representations* (1) *of which satisfy* (10) *and the supporting functions* (2) *of which satisfy* (17). *Let $S$ be the convex hypersurface represented by* (11), *let $C$ be the closed convex set determined by $S$ and let $U$ be the* (*possibly empty*) *open convex subset of $C$ determined by* (15). *Then the vectorial sum $T$ of $S_1, S_2, \cdots$ is $C - U$.*

The supporting function $h_R(\omega)$ of the boundary $R$ of $U$ satisfies the inequality

$$h_R(\omega) \leqq h_1(\omega) - \sum_{k>1} h_k(-\omega),$$

in view of the definition (15) of $U$. Moreover, if $\eta = \eta_0$ is any point of $R$ and $\omega = \omega_0$ is such that the equality sign holds in (18), then

$$(30) \qquad h_R(\omega_0) = h_1(\omega_0) - \sum_{k>1} h_k(-\omega_0)$$

in view of the expression (2) of the supporting function of a hypersurface.

**II. The hypersurfaces $S_E$.** Let again (9) be a sequence of convex hypersurfaces and suppose, for simplicity, that the functions (1) defining (9) are univalued functions of $\omega$, i. e. that any supporting hyperplane of $S_k$ has exactly one point in common with $S_k$; and that no degenerate $S_k$ occur, i. e. that in

(7), the equality sign is excluded. In what follows any sum of the form $\Sigma \eta_k(\omega)$ will denote the zero-vector if the set of subscripts $k$ over which the summation is taken is empty. Let $\epsilon_k$ $(k = 1, 2, \cdots)$, be $-1$, 0 or $+1$ and let $E$ be a symbol of the form

$$E \equiv (\epsilon_1, \epsilon_2, \cdots). \tag{31}$$

For a given symbol (31), let $S_E$ denote the locus represented by

$$S_E\colon \quad \eta = \eta_E(\omega) = \sum_{\epsilon_k \neq 0} \eta_k(\epsilon_k \omega). \tag{32}$$

The following remarks concerning these loci are quite obvious:

(i) The symbol obtained from $E$ by replacing every $\epsilon_k$ by $-\epsilon_k$ determines again the locus $S_E$. Thus all loci $S_E$ are obtained if one normalizes $E$ by the restriction that the first $\epsilon_k$ in $E$ which is not 0 shall be 1.

(ii) The locus (32) which corresponds to the symbol (31) in which $\epsilon_k = 1$ for every $k$, is the convex hypersurface denoted in Section I by $S$.

(iii) Similarly the locus (32) which corresponds to any symbol $E$ in which $\epsilon_k$ is either 0 or 1 for every $k$ is a convex hypersurface with the supporting function

$$h_E(\omega) = \sum_{\epsilon_k = 1} h_k(\omega) \tag{33}$$

(iv) If $E$ is any symbol (31), let $S_I$, $S_{II}$ be the convex hypersurfaces represented by

$$S_I\colon \quad \eta = \eta_I(\omega) = \sum_{\epsilon_k = 1} \eta_k(\omega); \qquad S_{II}\colon \quad \eta = \eta_{II}(\omega) = \sum_{\epsilon_k = -1} \eta_k(\omega). \tag{34}$$

Then $S_E$ may be represented in the form

$$S_E\colon \quad \eta = \eta_E(\omega) = \eta_I(\omega) + \eta_{II}(-\omega). \tag{35}$$

(v) The set of all symbols (31) may be made into a topological space $\mathcal{E}$ as follows: If an integer $n$, and $n$ numbers $\epsilon_1^0, \cdots, \epsilon_n^0$ equal to $-1$, 0, or $+1$ are given, the subset of $\mathcal{E}$ formed by those symbols $E$ for which $\epsilon_i = \epsilon_i^0$, $(i = 1, \cdots, n)$, is said to be open in $\mathcal{E}$. Also any sum of open sets in $\mathcal{E}$ is said to be open in $\mathcal{E}$. It is well known that the resulting topological space is homeomorphic with a Cantor set (except, of course, if the number of given convex hypersurfaces is finite). From the assumption (10) and the definition (32), it follows that $\eta_E(\omega)$ is a continuous function of the pair of variables $E, \omega$.

(vi) A symbol (31) is said to be of infinite length if $\epsilon_k \neq 0$ for every $k$. The subset of the topological space $\mathcal{E}$ consisting of all symbols of infinite

length clearly is compact and the $S_E$ corresponding to these symbols are subsets of the vectorial sum $T$ of all $S_k$. From the last remark in (v) it follows that the point set $F$ formed by the points of all these $S_E$ is a closed subset of $T$. The set $F$, which contains the boundary of $T$ by Section I, will be called the *irregular set* of $T$. The set $G = T - F$ is an open set and will be called the *regular set* of $T$. It may be seen from examples that $G$ may be empty and that $G$ may be dense in $T$.

(vii) If a symbol of infinite length is such that $\epsilon_k = -1$ for at least one $k$, then $S_E$ does not have a point in common with the exterior boundary of $T$. In fact for such an $E$, from (32) and (2), since the equality sign in (7) is excluded,

$$\omega \cdot \eta_E(\omega) = \sum_k \epsilon_k h_k(\epsilon_k \omega) < \sum_k h_k(\omega),$$

so that the statement follows from the remark made in connection with (8).

(viii) A symbol (31) is said to be of the finite length $l$ if $\epsilon_k \neq 0$ or $\epsilon_k = 0$ according as $k \leqq l$ or $k > l$. Clearly if $E$ is of length $l$, then $S_E$ is contained in the vectorial sum $T_l$ of $S_1, \cdots, S_l$. The irregular set $F_l$ of $T_l$ is formed by the $2^{l-1}$ curves $S_E$ corresponding to symbols (31) of length $l$.

In what follows use will be made of the following assumption, which will be referred to as condition (*):

(*) *The hypersurface $S_E$ which belongs to the symbol*

$$E = (+1, -1, -1, -1, \cdots) \tag{36}$$

*is convex and forms the interior boundary $R$ of the vectorial sum $T$ of the $S_k$.*

Concerning this condition the following remarks hold:

(ix) If condition (*) is satisfied, then the parameter representation $\eta_R(\omega)$ and the supporting function $h_R(\omega)$ of $R$ are given, in view of (30), by

$$\eta_R(\omega) = \eta_1(\omega) + \sum_{k>1} \eta_k(-\omega); \qquad h_R(\omega) = h_1(\omega) - \sum_{k>1} h_k(\omega). \tag{37}$$

(x) If condition (*) is satisfied and $E$ is any symbol (31) for which $\epsilon_1 = 1$, then $S_E$ is a convex hypersurface with the supporting function

$$\sum_{\epsilon_k \neq 0} \epsilon_k h_k(\epsilon_k \omega). \tag{38}$$

In fact, by (32) and (37), the parameter representation of $S_E$ may be brought into the form

$$\eta_E(\omega) = \eta_R(\omega) + \sum_{k>1,\, \epsilon_k=0} \{-\eta_k(-\omega)\} + \sum_{k>1,\, \epsilon_k=1} \{\eta_k(\omega) - \eta_k(-\omega)\}. \tag{39}$$

Since the separate terms in (39) are representations of convex hypersurfaces by (3), it follows, as in connection with (11) and (12), that (39) represents a convex hypersurface and that the supporting function of (39) is (38).

(xi) The following particular case of (x) is worth noting: If $E$ is a symbol (31) of length $l$, then $S_E$ is a convex hypersurface, moreover the hyperplane of normal direction $\omega$ through the point (32) of $S_E$ is a supporting plane of $S_E$.

(xii) If the condition in (xi) holds for every $l$, then the $S_E$ corresponding to any symbol $E$ of infinite length is a convex hypersurface, even if condition (*) is not satisfied. This follows easily from the last statement under (v).

(xiii) If (*) is satisfied and a symbol (31) is such that $\epsilon_k \neq 0$ for every $k$ and $\epsilon_k = 1$ for at least one $k > 1$, then the open convex region determined by the corresponding $S_E$ contains the interior boundary of $T$. In fact if $h_E(\omega)$ is the supporting function of $S_E$, then by (39)

$$h_E(\omega) - h_R(\omega) = \sum_{k > 1, \epsilon_k = 1} \{h_k(\omega) + h_k(-\omega)\} > 0,$$

since the equality sign is excluded in (7), so that the statement follows from the remark in connection with (8).

**III. A further consideration of the sets *F* and *G*** [II (vi)]. In what follows use will be made of the following remark. If $\eta = \eta_1(\omega)$ is the representation (1) of a convex hypersurface $S_1$ and $H$ represents a sphere of radius $r < \rho$ and centre at the origin together with its interior, then the vectorial sum of $S_1$ and $H$ does not contain the interior of any hypersphere of radius $\rho$ which does not meet $S_1$. This is clear since the distance of any part of this vectorial sum to the nearest point of $S_1$ is less than $\rho$.

Now suppose that (9) is a sequence of convex hypersurfaces satisfying (10), (17) and the conditions formulated at the beginning of Section II. Suppose in addition that the origin of the vector space is in the interior of each of the curves $S_k$, i. e., that

$$\omega \cdot \eta_k(\omega) = h_k(\omega) > 0 \tag{40}$$

for every $k$ and every $\omega$, and also that the $S_E$ corresponding to (9) have property II, (xi) for every $l$. Let $V$ be a given component of the regular set $G$ [II, (vi)] of the vectorial sum $T$ of the $S_k$. Now there exists an integer $m$, which depends on $V$ and has the following properties:

(i) if $E$ is a symbol (31) such that the corresponding $S_E$ contains a point of the boundary $V$, then the $\epsilon_k$ in $E$ for which $k > m$ are all equal.

(ii) for every $n > m$, the regular set $G_n$ of the vectorial sum $T_n$ of $S_1, \cdots, S_n$ has a component $V_n$ which contains $V$. Moreover $V_n$ contains $V_{n+1}$ for every $n > m$.

It will be shown first that if $\rho > 0$ is such that $V$ contains the interior of a hypersphere of radius $\rho$, and $m$ is such that

$$a_k < \tfrac{1}{2}\rho \text{ for every } k > m, \tag{41}$$

where $a_k$ is defined in (10), then $m$ satisfies (i). In fact let $E$ be a symbol (31) such that $\epsilon_p = -1$, $\epsilon_q = 1$, where $p$ and $q$ are fixed numbers larger than $l$, then the loci $S_+$, $S_E$, $S_-$ determined by

$$\eta = \eta_E(\omega) + \eta_p(\omega) - \eta_p(-\omega), \quad \eta = \eta_E(\omega), \quad \eta = \eta_E(\omega) - \eta_q(\omega) + \eta_q(-\omega)$$

are convex hypersurfaces contained in $F$ [II, (vi)]. Now it is clear from (7) and (41) that the remark at the beginning of this Section may be applied. This shows that a region which contains the interior of a hypersphere of radius $\rho$ and which does not meet $S_E$ but has a boundary point in $S_E$, must necessarily meet either $S_+$ or $S_-$. Thus $V$ cannot have a boundary point on $S_E$ and so an integer $m$ which satisfies (41) has property (i).

Next it will be shown that an integer $m$ has not only property (i) but also property (ii) if the numbers $a_k$ occurring in (10) satisfy not only (41) but also

$$\sum_{k>m} a_k < \tfrac{1}{4}\rho, \tag{42}$$

where $\rho$ is again the radius of a sphere, the interior of which is in $V$. Let $E$ be any symbol (31) of length $l > m$ and let $E_+$, $E_-$ be the symbols obtained from $E$ by replacing all $\epsilon_k$ for which $k > l$ (which are 0 by II, (viii)) by $+1, -1$ respectively. If $\eta_+(\omega)$, $\eta_-(\omega)$ are the parameter representations (32) of the hypersurfaces $S_+$, $S_-$ which correspond to $E_+$, $E_-$, then it follows from (40), (42) and (10) that $\eta_+(\omega) - \eta_-(\omega)$ (which is by (32), (3) and II, (iii) the representation (1) of some convex hypersurface) satisfies

$$0 < \omega \cdot (\eta_+(\omega) - \eta_-(\omega)) < \tfrac{1}{2}\rho.$$

It follows from the remark at the beginning of this Section that if the interior of some sphere of radius $\rho$ does not meet $S_+$ or $S_-$, then it does not meet $S_E$. Thus if $S_E$ is of length $l > m$, then $S_E$ does not meet $V$ and accordingly the region $V_l$ as defined in (ii) must exist if $l > m$. Obviously $V_l > V_{l+1}$, $l > m$, follows by the same argument, since no essential use has been made of the fact that the number of hypersurfaces in (9) is infinite.

Thus a number $m$ exists which has both property (i) and property (ii). Of course the results of this Section hold if condition (*) is satisfied.

**IV. The density of convolutions of distribution functions corresponding to convex curves.** From now on convex curves $S_k$ in the $(x, y)$-plane will be considered. Let the convex curve $S_k$ be given by

$$S_k:\quad x = x_k(\theta), \qquad y = y_k(\theta) \tag{43}$$

where $\theta$ is an angular parameter and where exactly one point of $S_k$ corresponds to each value of $\theta$. It is well known that (43) determines a distribution function $\phi_k(E)$, where $E$ is any Borel set in the $(x, y)$-plane, by the rule that $\phi_k(E)$ is the $\theta$ measure of the set of values of $\theta$ for which the point (43) of $S_k$ is in $E$. Obviously $S_k$ is the spectrum of $\phi_k$. The convolution

$$\psi_n = \phi_1 * \phi_2 * \cdots * \phi_n, \qquad n > 1 \tag{44}$$

has as its spectrum the vectorial sum $T_n$ of $S_1, \cdots, S_n$. On the assumption that (43) has a continuous derivative and that

$$x'_k(\theta)^2 + y'_k(\theta)^2 \neq 0 \text{ for every } \theta, \tag{45}$$

and finally that no $S_k$ contains a line segment, it has been proved [10] that the convolution of at least two $\phi_k$ is absolutely continuous and that the convolution of at least four $\phi_k$ has a continuous density. If it is also assumed that the functions (43) are regular analytic, then it has been shown [10] that the density of the convolution of two $\phi_k$ is a regular analytic function of $x$ and $y$ on the regular set of the vectorial sum of the corresponding $S_k$ (cf. (vi) in Section II) and that this density is not bounded in the vicinity of any point of the irregular set of the vectorial sum. This statement forms the special case where $n = 2$ of the following

THEOREM $2_n$. *Let the convex curves $S_k$ $(k = 1, 2, \cdots)$, in the $(x, y)$-plane be given by the regular analytic parameter representation* (43) *satisfying* (45) *and let the corresponding $S_E$ have property* (xi) *of Section II for $l < n$. Then the density $\delta_n(x, y)$ of* (44) *is regular analytic on the regular set $G_n$ of the vectorial sum $T_n$ of $S_1, \cdots, S_n$.*

It will be assumed that the following statement has been proved:

$(\S_n)$ On the assumptions of Theorem $2_n$, not only the statement of the Theorem holds but also: If $P$ is a common point of exactly $p$ distinct curves $S_E$, where $E$ is of length $n$, and $S^i$ $(i = 1, \cdots, p)$, denotes these curves, then a vicinity $U$ of $P$ may be found, and a regular analytic function $\lambda_i(x, y)$ in the complement of $S^i$ in $U$ $(i = 1, \cdots, p)$, such that for any point $(x, y)$ in $U$,

---

[10] Van Kampen and Wintner [6], p. 103.

(46) $$\delta_n(x,y) = \sum_{i=1}^{p} \lambda_i(x,y).$$

Needless to say $\lambda_i(x,y)$ consists in general of distinct regular functions in the parts into which $U$ is divided by $S^i$. Thus (46) is to the effect that singularities of $\delta_n(x,y)$ at the several curves $S_E$ are added at common points of distinct curves $S_E$. In case $n=2$, Theorem $2_n$ and $(\S_n)$ are correct, the latter by (vii) of Section II. It will be shown that $(\S_n)$ implies $(\S_{n+1})$, thus completing the proof by complete induction of Theorem $2_n$.

If the conditions of Theorem $2_{n+1}$ are satisfied, then clearly $F_n$ consists of $2^{n-1}$ convex analytic curves $S_E$, so that any singularity of $F_n$ is of the type described in $(\S_n)$.

Since $\psi_{n+1} = \psi_n * \phi_{n+1}$, one has for the density $\delta_{n+1}(x,y)$ of $\psi_{n+1}$

$$\delta_{n+1}(x,y) = \int \delta_n(x-\xi, y-\eta)\phi_{n+1}(d\xi, \eta E)$$

over the $(\xi,\eta)$-plane, or, by the definition of $\phi_{n+1}(E)$,

(47) $$\delta_{n+1}(x,y) = \int \delta_n(x - x_{n+1}(\theta),\ y - y_{n+1}(\theta))d\theta$$

over all angles $\theta$. For a fixed $(x_0,y_0)$ let $P_1,\cdots,P_q$ be the distinct common points of $F_n$ and the curve

(48) $$x = x_0 - x_{n+1}(\theta), \qquad y = y_0 - y_{n+1}(\theta)$$

and let $I_1,\cdots,I_q$ be non-overlapping $\theta$-intervals containing $P_1,\cdots,P_q$ such that the arcs of (48) corresponding to these intervals are contained in the vicinities corresponding to $P_1,\cdots,P_q$ by $(\S_n)$. Clearly the contribution to (47) of the $\theta$-intervals, obtained by omitting $I_1,\cdots,I_q$ from the range of $\theta$, is regular in a vicinity of $(x_0,y_0)$. Suppose that (48) is not tangent to the curve $S^i$ of $F_n$ which passes through $P=P_j$. Then the contribution

(49) $$\int_{I_j} \lambda_i(x - x_{n+1}(\theta), y - y_{n+1}(\theta))d$$

to (47) obviously is regular in a vicinity of $(x_0,y_0)$ in view of $(\S_n)$. Now suppose that (48) is tangent to the curve $S^i$ of $F_n$ at $P=P_j$. Then clearly (49) is regular in a vicinity of $(x_0,y_0)$ except at those points $x$, $y$ of this vicinity for which the curve corresponding to (48) still is tangent to $S^i$. Thus $(\S_{n+1})$ will follow if it is shown that to a point of tangency $P_j$ of (48) and $S^i$ there corresponds a point of one of the curves of $F_{n+1}$ at $(x_0,y_0)$. Now, if $\omega'$ is the value of $\omega$ corresponding to the point $P_j$ of $S^i$, then the corre-

sponding point of $S_{n+1}$ belongs to $\omega'$ or to $-\omega'$, so that the statement follows from the definition (32) of the curves $S_E$ constituting $F_{n+1}$.

*Remark.* In Theorem $2_n$ it may be allowed that one or more of the curves $S_E$ occurring degenerates into a single point. In that case "a curve is tangent to $S_E$" should be read "the curve passes through the point into which $S_E$ degenerates."

Now let the convex curves $S_k$ given in the $z$-plane be infinite in number and let (10) be satisfied. Then [11] the infinite convolution

$$\psi = \lim \psi_n = \phi_1 * \phi_2 * \phi_3 * \cdots \tag{50}$$

is convergent and has as its spectrum the vectorial sum $T$ of the $S_k$, while

$$\delta(x, y) = \lim \delta_n(x, y), \tag{51}$$

where $\delta(x, y)$ is the density of (50).

THEOREM 3. *Let the curves $S_k$ be given by the regular analytic representations (43) satisfying (45) and let the corresponding $S_E$ have property (xi) of Section II for every $l$. Then the density (51) of (50) is regular analytic on the regular set $G$ of the vectorial sum $T$ of the $S_k$.*

In fact, let $(x_0, y_0)$ be any point of $G$, let $3\rho > 0$ be less than the distance from $(x_0, y_0)$ to the irregular set in $T$ and let $N$ be an integer such that

$$\sum_{n=N+1}^{\infty} a_n < \rho,$$

where the $a_n$ are the numbers occurring in (10). Then clearly the circular disk with radius $2\rho$ and centre at $(x_0, y_0)$ belongs to the regular set $G_N$ of $T_N$ and the spectrum of

$$\psi^N = \phi_{N+1} * \phi_{N+2} * \cdots \tag{52}$$

is within a distance $\rho$ from the origin. If $\delta^N(x, y)$ is the density of $\psi^N$, then

$$\delta(x, y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \delta_N(x - \xi, y - \eta)\delta^N(\xi, \eta)\, d\xi\, d\eta \tag{53}$$

since $\phi = \psi_N * \psi^N$. This integral need only be taken over the spectrum of $\psi^N$, since $\delta^N(\xi, \eta) = 0$ unless $(\xi, \eta)$ is in this spectrum. On the other hand if $(x, y)$ is within a distance $\rho$ of $(x_0, y_0)$ and $(\xi, \eta)$ is in the spectrum of $\psi^N$, then $(x - \xi, y - \eta)$ is within a circle of radius $2\rho$ and centre at $(x_0, y_0)$.

[11] *Ibid.*, pp. 184-186.

Thus if $(x, y)$ is in a circle of radius $\rho$ and centre at $(x_0, y_0)$, then the integrand of (53) is a regular analytic function of $(x, y)$, so that $\delta(x, y)$ is a regular analytic function of $(x, y)$. This completes the proof of Theorem 3.

Clearly in Theorem 3 the condition that II, (xi) is satisfied for every $l$ may be replaced by the condition that the $S_k$ satisfy the condition (*) introduced in II.

**V. The logarithm of the Riemann zeta function.** If $p_m$ denotes the $m$-th prime number it is known[12] that for a fixed $\sigma > 1$, the asymptotic distribution function of the almost periodic function

$$(54) \qquad \begin{aligned} f_\sigma(t) &= -\log \zeta(\sigma + it) + \tfrac{1}{2} \log \zeta(2\sigma) \\ &= \sum_{m=1}^{\infty} [\log (1 - p_m{}^{-\sigma-it}) - \tfrac{1}{2} \log (1 - p_m{}^{-2\sigma})] \end{aligned}$$

may be represented as the infinite convolution

$$\phi^\sigma = \phi_1{}^\sigma * \phi_2{}^\sigma * \cdots,$$

where $\phi_m{}^\sigma$ denotes the distribution function which belongs to the convex curve (43) defined by

$$(55) \qquad S_m{}^\sigma: \quad x + iy = \log (1 - p_m{}^{-\sigma} e^{i\theta}) - \tfrac{1}{2} \log (1 - p_m{}^{-2\sigma})$$

or in other words by

$$(56) \qquad x = x_m{}^\sigma = r(\theta, p_m{}^{-\sigma}), \qquad y = y_m{}^\sigma = s(\theta, p_m{}^{-\sigma}),$$

where

$$(57) \qquad \begin{aligned} S_\rho: \quad x = r(\theta, \rho) &= \tfrac{1}{2} \log \frac{1 - 2\rho \cos \theta + \rho^2}{1 - \rho^2}, \\ y = S(\theta, \rho) &= \arctan \frac{\rho \sin \theta}{1 - \rho \cos \theta} \end{aligned}$$

is the equation of a system of curves $S_\rho$ depending continuously on a parameter $\rho$, it being understood that the arc tan takes only values between $-\tfrac{1}{2}\pi$ and $+\tfrac{1}{2}\pi$. For $0 < \rho < 1$, $S_\rho$ represents a convex analytic curve satisfying (45) and having both axes $x = 0$, $y = 0$ as lines of symmetry.

Let $h(\rho, \omega)$ denote the supporting function of $S_\rho$, where now $\omega$ denotes not as in I, II, III, a variable unit vector in the $(x, y)$-plane, but the angle between such a vector and the positive $x$-axis. For reasons of symmetry it is sufficient to consider only the interval $0 \leqq \omega \leqq \tfrac{1}{2}\pi$. Clearly

$$(58) \qquad \begin{aligned} h(\rho, 0) &= \tfrac{1}{2} \log [(1 + p_m{}^{-\sigma})/(1 - p_m{}^{-\sigma})], \\ h(\rho, \tfrac{1}{2}\pi) &= \arcsin \rho^{-\sigma} (\leqq \tfrac{1}{2}\pi). \end{aligned}$$

[12] Van Kampen and Wintner [6], Section 7, where further references are given.

Now let $\rho_1, \rho_2, \cdots$ be a sequence of numbers such that $\rho_1 < 1$ and $0 < \rho_{n+1} < \rho_n$ and that $\sum_n h(\rho_n, \omega)$ converges uniformly in $\omega$. It will be shown that the function

(59) $$f(\omega) = h(\rho_1, \omega) - \sum_{n>2} h(\rho_n, \omega)$$

has the following properties:

(60 a) $f(\omega) > 0$ for every $\omega$, in case $h(\rho_1, \frac{1}{2}\pi) > \sum_{n>2} h(\rho_n, \frac{1}{2}\pi)$,

(60 b) $f(\omega) < 0$ for every $\omega$, in case $h(\rho_1, 0) < \sum_{n>2} h(\rho_n, 0)$,

(60 c) $f(\omega_0) = 0$, for an $\omega = \omega_0$, $f(\omega) > 0$ if $0 \leqq \omega < \omega_0$, $f(\omega) < 0$ if $\omega_0 < \omega \leqq \frac{1}{2}\pi$, in case $h(\rho_1, \frac{1}{2}\pi) \leqq \sum_{n>2} h(\rho_n, \frac{1}{2}\pi)$ and $h(\rho_1, 0) \geqq \sum_{n>2} h(\rho_n, 0)$.

Clearly (60 a), (60 b), (60 c) follow from the following statement

(61) $$\frac{h(\rho_1, \omega_1)}{h(\rho_1, \omega_2)} < \frac{h(\rho_2, \omega_1)}{h(\rho_2, \omega_2)} \text{ if } 0 < \rho_1 < \rho_2 < 1,\ 0 \leqq \omega_1 < \omega_2 \leqq \tfrac{1}{2}\pi.$$

For the proof of (61) use will be made of the following properties of the curves $S^\rho$:[13]

Let $S_\rho$ denote the curve similar to $S^\rho$ with respect to the origin in ratio $1/(\arcsin r)$, so that, by (58), the curves $S_\rho$ have the points $(0, \pm 1)$ in common for $0 < \rho < 1$. Then $S_{\rho_1}$ is, except for those two points, in the interior of $S_{\rho_2}$, whenever $0 < \rho_1 < \rho_2 < 1$. Moreover, if $\alpha(\rho, s)$ denotes, for $0 < s < 1$, the angle between the line $y = s$ and the normal of $S_\rho$ at the intersection of this line and $S_\rho$, then $\alpha(\rho, s)$ is an increasing function of $\rho$ in $0 < \rho < 1$.

In proving (61) it is clearly admissible to replace $h(\rho, \omega)$ by the supporting function of any convex curve $S'_\rho$ similar to $S^\rho$ with respect to the origin. On choosing, for fixed values $\rho_1, \rho_2$ $(\rho_1 < \rho_2)$ and $\omega_1$, the curves $S'_\rho$ in such a way that $h(\rho_1, \omega_1) = h(\rho_2, \omega_2)$, it is clear from the geometrical properties of $S_\rho$ mentioned above that $h(\rho_2, \omega_2) < h(\rho_1, \omega_1)$ if $\omega_2 > \omega_1$ is sufficiently near to $\omega_1$. Thus $h(\rho_1, \omega)/h(\rho_2, \omega)$ is an increasing function of $\omega$ if $0 < \rho_1 < \rho_2 < 1$ and (61) is proved.

Let $\sigma^k$, $\sigma_k$ denote, for any positive integer $k$, the obviously unique numbers such that

(62 a) $$\arcsin p_k^{-\sigma} = \sum_{m>k} \arcsin p_m^{-\sigma} \text{ if } \sigma = \sigma^k > 1$$

(62 b) $$\tfrac{1}{2}\log\left[(1 + p_k^{-\sigma})/(1 - p_k^{-\sigma})\right] = \sum_{m>k} \tfrac{1}{2}\log\left[(1 + p_m^{-\sigma})/(1 - p_m^{-\sigma})\right] \text{ if } \sigma = \sigma_k > 1$$

[13] Bohr and Jessen [1]. In this paper (61) is proved for the case $\omega_2 = \frac{1}{2}\pi$. The general proof given below is an extension of the proof given there.

and let $\bar{\sigma}$ denote the number such that

(62 c) $$p_1^{-\sigma} = \sum_{m>1} p_m^{-\sigma}, \text{ if } \sigma = \bar{\sigma} > 1.$$

Then it is clear from (60 a), (60 b), (60 c) that $\sigma^k > \sigma_k$ for every $k > 0$, while the numerical values $\sigma^1 = 1.764\ldots$ and $\bar{\sigma} = 1.778\ldots$ show that $\sigma_1 < \sigma^1 < \bar{\sigma}$. Clearly for sufficiently large $k$, $\sigma^k > \sigma_k > \bar{\sigma}$.

Consider now the case [14] where for some integer $l$,

(63) $$\sigma > \operatorname{Max}(\bar{\sigma}^{(1)}, \cdots, \sigma^{(l)}).$$

Let $E_I$ and $E_{II}$ be the symbols (31) obtained from a given symbol of length $l-1$ by replacing $\epsilon_l$ by $+1$, $\epsilon_k$ by $-1$, $k > l$, and $\epsilon_l$ by $-1$, $\epsilon_k$ by $+1$, $k > l$, respectively. Let $S_I$, $S_{II}$ be the corresponding curves (32), $\eta_I(\omega)$, $\eta_{II}(\omega)$ their parametric representations and $h_I(\omega)$, $h_{II}(\omega)$ their supporting functions. It is known [15] that, whenever $\sigma \geqq \bar{\sigma}$, the vectorial sum of the $S_m^\sigma$ satisfies condition (*), so that, by II (x), the curves $S_E$ corresponding to the $S_m^\sigma$ are convex curves and Theorem 3 is applicable. Now the functions $h_I$, $h_{II}$ satisfy, by (32) and the definition of $h(\rho, \omega)$, the equality

(64) $$h_I(\omega) - h_{II}(\omega) = 2h(p_l^{-\sigma}, \omega) - 2\sum_{m>l} h(p_m^{-\sigma}, \omega).$$

Thus it follows from (60 a), (58) and the definition (62 a) of $\sigma^k$ that $h_I(\omega) - h_{II}(\omega) > 0$. Hence $S_I$ and $S_{II}$ are exterior and interior boundary of a ringshaped subset of the vectorial sum of all $S_m^\sigma$. In a similar way it may be shown that in view of (63) no $S_E$ corresponding to a symbol $E$ of infinite length does enter the ring shaped region determined by $S_I$ and $S_{II}$. Thus this ring shaped region is by Theorem 3 a region of regular analyticity for the density of $\phi^\sigma$.

A region of regular analyticity as found above does not disappear abruptly if $\sigma$ does not satisfy (63) but is very near to values satisfying (63). In fact, since all curves $S_E$ depend on $\sigma$ in a uniformly continuous way, if $\sigma$ is restricted to a bounded interval with positive endpoints, it is clear that the ring shaped region described above remains ring shaped for certain $\sigma$ if $\sigma > \operatorname{Max}(\bar{\sigma}, \sigma^l)$ remains satisfied, but $\sigma$ becomes less than $\sigma^k$ for some $k = 1, \cdots, l-1$.[16] It will be shown below that the ring shaped region constructed above is replaced by a pair of crescent shaped regions if $\sigma$ satisfies

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[14] This case has already been considered by van Kampen and Wintner [6], Section 7.

[15] Bohr and Jessen [1]; Kershner [5].

[16] It seems to require elaborate numerical calculations to decide whether or not the sequences $\{\sigma^k\}$, $\{\sigma_k\}$ are monotone.

$$\sigma^k > \sigma > \operatorname{Max}(\bar{\sigma}, \sigma^1, \cdots, \sigma^{l-1}, \sigma_l) \tag{65}$$

In fact, if (65) is satisfied, then the difference (64) of $h_I(\omega)$ and $h_{II}(\omega)$ is a function satisfying (60 c), so that the curves $S_I$ and $S_{II}$ determine four crescent shaped regions, two of which are halved by the $x$-axis and two of which are halved by the $y$-axis. It is clear from (60 c) that the first two regions have $S_I$ as exterior and $S_{II}$ as interior boundary, while the opposite is true for the last two regions. Since $S_I$ is in the interior of the curves $S_E$ corresponding to symbols $E$ of infinite length adjacent to $E_I$, and $S_{II}$ is exterior to the curves $S_E$ corresponding to symbols $E$ of infinite length adjacent[17] to $E_{II}$, it is clear that the last two regions are not contained in the regular set of the vectorial sum of all $S_m{}^\sigma$. On the other hand, it may be inferred easily from (65) that no $S_E$ corresponding to symbols $E$ of infinite length has a point in common with the two crescent shaped regions which are halved by the $x$-axis. Thus the latter two regions are regions of regular analyticity of the density of $\phi^\sigma$ by Theorem 3. It is clear that the two regions just described decrease in size as $\sigma$ decreases from $\sigma^l$ to $\sigma_l$ and disappear completely when $\sigma$ becomes equal to $\sigma_l$. The two regions may, of course, cease to be regions of regular analyticity of the density of $\phi^\sigma$ for some $\sigma > \sigma_l$ if for instance $\sigma^{l-1} > \sigma_l$, a case which probably occurs for sufficiently large $l$.

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[17] For the meaning of words like "adjacent" as applied to symbols (31) compare II, (v).

# ON THE PARTIAL SUMS OF CERTAIN FOURIER SERIES.*

By Otto Szász.

**1.** Suppose $f(x)$ is real-valued, periodic of period $2\pi$, and Lebesgue integrable over $(-\pi,\pi)$. Denote its Fourier series by

$$f(x) \sim a_0/2 + \sum_{\nu=1}^{\infty} (a_\nu \cos \nu x + b_\nu \sin \nu x), \tag{1}$$

and let its partial sums be

$$s_0 = a_0/2, \quad s_n = s_n(x) = a_0/2 + \sum_{\nu=1}^{n} (a_\nu \cos \nu x + b_\nu \sin \nu x), \quad (n = 1, 2, \cdots).$$

In 1932, Paley [3] proved the two theorems which follow:

I. *Suppose*

$$|f(x)| \leq M \text{ in } (-\pi,\pi), \tag{2}$$

$$a_0 \geq 0, \qquad a_n \geq 0, \qquad b_n \geq 0, \qquad (n = 1, 2, \cdots), \tag{3}$$

*then*

$$|s_n(x)| \leq 10M, \qquad -\pi \leq x \leq \pi, \qquad (n = 0, 1, 2, \cdots). \tag{4}$$

II. *If in addition $f(x)$ is continuous, then its Fourier series* (1) *converges to $f(x)$ uniformly for all $x$.*

For Theorem I Fejér [1] gave a simpler proof and replaced the constant 10 by 4, so that

$$|s_n(x)| \leq 4M. \tag{5}$$

I give in (§ 2) another elementary proof for both theorems simultaneously. At the same time I replace (5) by (§§ 3-4)

$$|s_n(x)| \leq M(2 + \alpha/\sin^2 \alpha) < 3.38M, \tag{6}$$

where $\alpha$ is the unique root of the equation $2\alpha = \tan \alpha$ in the interval $0 < \alpha < \pi/2$.

If $\beta$ is the least number such that (2) and (3) imply

$$|s_n(x)| \leqq \beta M, \qquad -\pi \leq x \leq \pi, \qquad (n = 0, 1, 2, \cdots),$$

* Received April 26, 1937.

then, by (6),

$$\beta \leqq 2 + \alpha/\sin^2 \alpha < 3.38.$$

On the other hand if $f(x) = \pi - x/2$, $0 < x < \pi$, $f(-x) = -f(x)$, then

$$f(x) = \sum_1^\infty \frac{\sin \nu x}{\nu},$$

$$M = \pi/2, \text{ and } s_n(\pi/n+1) \uparrow \int_0^\pi \frac{\sin x}{x}\, dx = 1.851 \cdots;$$

hence

$$\beta \geqq \frac{2}{\pi} \int_0^\pi \frac{\sin x}{x}\, dx > 1.17.$$

Thus $\beta$ lies somewhere between 1.17 and 3. 38.

In §§ 5-6, I consider generalized trigonometric series, improving on a theorem by M. Fekete [2].

**2.** On putting

$$\frac{f(x)+f(-x)}{2} = \phi(x), \quad \int_0^x \phi(t)\,dt = \phi_1(x), \quad \int_0^x \phi_1(t)\,dt = \phi_2(x),$$

we have

$$\phi(x) \sim \frac{a_0}{2} + \sum_{\nu=1}^\infty a_\nu \cos \nu x$$

$$\phi_1(x) = \tfrac{1}{2} a_0 x + \sum_{\nu=1}^\infty \frac{a_\nu}{\nu} \sin \nu x$$

$$\begin{aligned}\phi_2(x) &= \tfrac{1}{4} a_0 x^2 + \sum_{\nu=1}^\infty \frac{a_\nu}{\nu^2}(1-\cos \nu x) = \tfrac{1}{4} a_0 x^2 + 2\sum_{\nu=1}^\infty \frac{a_\nu}{\nu^2} \sin^2 \tfrac{1}{2}\nu x \\ &\geq a_0 \left(\frac{x}{2}\right)^2 + 2\left(\frac{x}{2}\right)^2 \sum_{\nu=1}^n a_\nu \left(\frac{\sin \frac{1}{2}\nu x}{\frac{1}{2}\nu x}\right)^2, \qquad (n=1,2,\cdots),\end{aligned}$$

if $a_n \geq 0$, $(n=1,2,\cdots)$. We now assume in addition, instead of (2),

$$\phi(x) \leqq M \text{ in } (-\pi, \pi).$$

Hence, a fortiori, $\phi_2(x) \leq x^2 M/2$ in $0 < x < \pi$, and

$$\frac{a_0}{2} + \sum_{\nu=1}^n a_\nu \left(\frac{\sin \frac{1}{2}\nu x}{\frac{1}{2}\nu x}\right)^2 \leq M, \qquad (n=1,2,3,\cdots). \tag{7}$$

Letting now $x \downarrow 0$, (7) takes the form

$$\frac{a_0}{2} + \sum_{\nu=1}^n a_\nu \leq M, \qquad (n=1,2,\cdots). \tag{8}$$

Hence $\sum a_\nu$ converges, and $\sum a_\nu \cos \nu x$ converges uniformly in any interval. Moreover, by (8),

$$\sum_{\nu=1}^{n} a_\nu \leq M - \frac{a_0}{2}; \text{ whence } \sum_{\nu=1}^{n} a_\nu \mid \cos \nu x \mid \leq M - \frac{a_0}{2},$$

and

$$\frac{a_0}{2} + \sum_{\nu=1}^{n} a_\nu \cos \nu x \leq \frac{a_0}{2} + \sum_{1}^{n} a_\nu \mid \cos \nu x \mid \leq M. \tag{9}$$

Furthermore, let $\phi(x) \geqq -M$; then from $\frac{a_0}{2} = \frac{1}{\pi}\int_0^\pi \phi(x)\,dx$ it follows that $-2M \leqq a_0 \leq 2M$. Also by (8)

$$\tfrac{1}{2}a_0 + \sum_{1}^{n} a_\nu \cos \nu x \geq -\sum_{1}^{n} a_\nu \geq \tfrac{1}{2}a_0 - M \geq -M. \tag{10}$$

On putting

$$\frac{f(x) - f(-x)}{2} = \psi(x), \qquad \int_0^x \psi(t)\,dt = \psi_1(x),$$

we have

$$\psi(x) \sim \sum_{1}^{\infty} b_\nu \sin \nu x,$$

$$\frac{1}{x}\psi_1(x) = 2\sum_{1}^{\infty} b_\nu \frac{\sin^2 \frac{1}{2}\nu x}{\nu x} \geq 2\sum_{1}^{n} b_\nu \frac{\sin^2 \frac{1}{2}\nu x}{\nu x} = \frac{x}{2}\sum_{1}^{n} \nu b_\nu \left(\frac{\sin \frac{1}{2}\nu x}{\frac{1}{2}\nu x}\right)^2.$$

Since $\sin u/u$ is monotonic decreasing in $0 < u < \pi/2$, this becomes for $x = \pi/n$

$$\frac{4}{\pi^2} \cdot \frac{\pi}{2n}\sum_{1}^{n} \nu b_\nu \leq \frac{n}{\pi}\psi_1\left(\frac{\pi}{n}\right), \qquad (n = 1, 2, 3, \cdots),$$

or

$$\frac{2}{n\pi}\sum_{1}^{n} \nu b_\nu \leq \frac{n}{\pi}\psi_1\left(\frac{\pi}{n}\right), \qquad (n = 1, 2, 3, \cdots), \tag{11}$$

If now $\psi(x) \leq M$ in $\mid x \mid < \pi$, and a fortiori $\psi_1(x)/x \leq M$, $0 < x < \pi$, then (11) leads to

$$\frac{2}{n\pi}\sum_{1}^{n} \nu b_\nu \leq M, \qquad \frac{1}{n}\sum_{1}^{n} \nu b_\nu \leq \frac{\pi}{2}M, \qquad (n = 1, 2, 3, \cdots). \tag{12}$$

If, in addition, $\psi(x)$ is continuous at $x = 0$, then $\psi_1(x)/x \to 0$ for $x \downarrow 0$, and from (11) we derive

$$\sum_{1}^{n} \nu b_\nu = o(n), \qquad n \to \infty. \tag{13}$$

Using the identity

$$U_n = \frac{1}{n}\sum_{1}^{n-1} U_\nu + \frac{1}{n}\sum_{1}^{n} \nu u_\nu, \qquad U_n = \sum_{1}^{n} u_\nu, \qquad (n = 2, 3, \cdots), \tag{14}$$

where we put $u_n = b_n \sin nx$, Fejér's theorem on the arithmetic means and (12) lead to

$$(15) \qquad |\sum_1^n b_\nu \sin \nu x| \leq M + \frac{\pi}{2} M = \left(1 + \frac{\pi}{2}\right) M;$$

(9), (10) and (15) yield Theorem I with the smaller constant $2 + \pi/2$ instead of 4 in the inequality (5).

Moreover, if $f(x)$ is continuous throughout, Theorem II follows from (13) and (14).

**3.** We shall generalize the assumptions on $\phi(x)$ and $\psi(x)$, and improve upon the constant in (12). We first prove

THEOREM 1. *Let* $a_\nu \geq 0$, $(\nu = 0, 1, 2, \cdots)$, *and* $\sum_1^\infty \nu^{-2} a_\nu < \infty$; *then*

$$(16) \qquad \frac{a_0}{2} + \sum_{\nu=1}^\infty \left(\frac{\sin \nu x}{\nu x}\right)^2 a_\nu = R_1(x)$$

*exists for* $x > 0$. *Suppose moreover*

$$\liminf_{x \to 0} R_1(x) = M,$$

*then*

$$\tfrac{1}{2}a_0 \leq M, \quad |\tfrac{1}{2}a_0 + \sum_1^n a_\nu \cos \nu x| \leq M, \quad |x| < \pi, \qquad (n = 1, 2, \cdots),$$

*and* $\Sigma a_\nu \cos \nu x$ *converges uniformly on the real axis.*

For the proof, note that (16) implies

$$\frac{a_0}{2} + \sum_1^n \left(\frac{\sin \nu x}{\nu x}\right)^2 a_\nu \leq R_1(x), \qquad (n = 1, 2, 3, \cdots);$$

let $x_k \downarrow 0$ and $\lim_{k \to \infty} R_1(x_k) = M$; then

$$\lim_{k \to \infty} \left\{ \frac{a_0}{2} + \sum_1^n \left(\frac{\sin \nu x_k}{\nu x_k}\right)^2 a_\nu \right\} \leq \lim_{k \to \infty} R_1(x_k) = M.$$

Hence,

$$\frac{a_0}{2} + \sum_1^n a_\nu \leq M, \quad (n = 1, 2, \cdots), \text{ and } \tfrac{1}{2}a_0 \leqq M.$$

The theorem follows at once from this.

If in particular $\phi(x) \sim \frac{1}{2}a_0 + \sum_1^\infty a_\nu \cos \nu x$, then

$$R_1(\tfrac{1}{2}x) = \frac{2}{x^2} \int_0^x \phi(t)(x - t)\, dt,$$

and we have the

COROLLARY. *If*

$$\phi(x) \sim \tfrac{1}{2}a_0 + \sum_1^\infty a_\nu \cos \nu x, \qquad a_\nu \geqq 0, \qquad (\nu = 0, 1, 2, \cdots),$$

*and*

$$\liminf_{x\to\infty} \frac{2}{x^2}\int_0^x \phi(t)(x-t)\,dt = M,$$

*then* $\sum_1^\infty a_\nu < \infty$ *and* $\tfrac{1}{2}a_0 + \sum_1^\infty a_\nu \leqq M$.

We call $u_0 + \sum_1^\infty \left(\frac{\sin \nu x}{\nu x}\right)^2 u_\nu$ the Riemannian mean of the first kind corresponding to the series $\sum_1^\infty u_\nu$.

**4.** We next prove

THEOREM 2. *Let* $b_\nu \geqq 0$, $(\nu = 1, 2, 3, \cdots)$, *and* $\sum_1^\infty \nu^{-1} b_\nu < \infty$; *then*

$$\frac{2x}{\pi}\sum_1^\infty \nu b_\nu \left(\frac{\sin \nu x}{\nu x}\right)^2 = R_2(x) \tag{17}$$

*exists for* $x > 0$. *Suppose, in addition,* $R_2(x) \leqq 2M/\pi$ *in an interval* $0 < x \leqq \delta$; *then*

$$\sum_1^n \nu b_\nu \leqq \frac{\alpha}{\sin^2 \alpha} Mn < 1.38Mn \quad \text{for} \quad n \geqq \frac{\alpha}{\delta}, \tag{18}$$

*where* $\alpha$ *is the unique root of the equation* $2\alpha = \tan \alpha$ *in* $0 < \alpha < \pi/2$.

We have from (17)

$$R_2(x) \geqq \frac{2x}{\pi}\sum_1^n \nu b_\nu \left(\frac{\sin \nu x}{\nu x}\right)^2 \geqq \frac{2x}{\pi}\left(\frac{\sin nx}{nx}\right)^2 \sum_1^n \nu b_\nu$$

for $0 < nx \leqq \pi/2$; or

$$\frac{1}{n}\sum_1^n \nu b_\nu \leqq \frac{\pi}{2}\,\frac{nx}{\sin^2 nx} R_2(x) \leqq M\frac{nx}{\sin^2 nx} \leqq M\frac{\alpha}{\sin^2 \alpha}, \qquad n \geqq \alpha/\delta, \tag{19}$$

where $\alpha/\sin^2 \alpha$ is the minimum of $t/\sin^2 t$ in $0 < t < \pi/2$; $\alpha$ is easily found to be the unique root of the equation $2\alpha = \tan \alpha$ in $0 < \alpha < \pi/2$. The "Table of Functions" by Jahnke and Emde (2nd edition, 1933, p. 33) gives $1.16 < \alpha < 1.17 = \alpha_1$, and $\sin \alpha_1/\alpha_1 = 0.7870 \cdots$. Then by a simple calculation

$$\frac{\alpha_0}{\sin^2 \alpha_0} < \frac{\alpha_1}{\sin^2 \alpha_1} < 1.38.$$

This proves the theorem.

If in particular $\psi(x) \sim \sum_1^\infty b_\nu \sin \nu x$ and $|\psi(x)| \leqq M$, then relations (18) and (14), with $u_\nu = b_\nu \sin \nu x$, give

$$\left|\sum_1^n b_\nu \sin \nu x\right| \leq M\left(1 + \frac{\alpha}{\sin^2 \alpha}\right) < 2.38M.$$

Furthermore

$$\frac{1}{x}\int_0^x \psi(t)\,dt = \sum_1^n b_\nu \frac{1-\cos \nu x}{\nu x} = \frac{x}{2}\sum_1^\infty \nu b_\nu \left(\frac{\sin \frac{1}{2}\nu x}{\frac{1}{2}\nu x}\right)^2 = \frac{\pi}{2} R_2\left(\frac{x}{2}\right);$$

and, from (19),

$$\frac{1}{n}\sum_1^n \nu b_\nu \leq \frac{\alpha}{\sin^2 \alpha}\cdot\frac{1}{x}\int_0^x \psi(t)\,dt, \qquad 0 < nx \leq \pi;$$

in particular, if $\frac{1}{x}\int_0^x \psi(t)\,dt \to 0$ as $x \to 0$, then

$$\frac{1}{n}\sum_1^n \nu b_\nu \to 0 \text{ for } n \to \infty.$$

We call $\frac{2x}{\pi}\left\{ s_0 + \sum_1^\infty \left(\frac{\sin \nu x}{\nu x}\right)^2 s_\nu \right\}$ the Riemannian mean of the second kind corresponding to the sequence $\{s_n\}$, or to the series $\sum_0^\infty u_\nu$ with $\sum_0^n u_\nu = s_n$.

**5.** We now pass to generalized trigonometric series and to almost periodic functions. The most general result in the case of positive coefficients is due to M. Fekete [2].

Let $\phi(x)$ be a measurable real function of the real variable $x$,

$$\sup_x \frac{1}{x}\int_0^x \phi^2(t)\,dt < \infty,$$

and let $\phi(-x) = \phi(x)$. Suppose that $\phi(x) \leqq U$, and that

$$\lim_{x\to\infty}\frac{1}{x}\int_0^x \phi(t)\cos \lambda t\,dt = a(\lambda) = M\{\phi(t)\cos \lambda t\} \tag{20}$$

exists for all $\lambda \geqq 0$. It is easy to show that $a(\lambda)$ vanishes, except at an enumerable set of $\lambda$-values. Denote, in a certain order, by $\lambda_1, \lambda_2, \cdots$ those $\lambda$ for which $a(\lambda) \neq 0$; we call the series

$$\sum_1^\infty a_n \cos \lambda_n x \sim \phi(x), \tag{21}$$

where $a_0 = a(0)$, $a_n = 2a(\lambda_n)$, in the Fourier expansion of $\phi(x)$.

Let $\rho_1, \rho_2, \cdots$ denote a subsequence of $\{\lambda_n\}$, consisting of linearly independent numbers; that is, no equation of the form $\sum_1^n r_\nu \rho_\nu = 0$ holds, where the $r_\nu$ are rational and not all zero. Denote by $\{\mu_n\}$ the subsequence of those $\lambda_n$, which can be represented in the form

$$\lambda_n = \sum_{\nu=1}^{h} r_\nu \rho_\nu, \qquad h = h(n); \qquad r_\nu = r_\nu(n) \text{ rational.}$$

Let $0 < \omega < \Omega$ denote given positive numbers. With this notation (with the restriction $|\phi(x)| \leq U$) Fekete proved the theorem:

*Suppose that*

$$a(\mu_n) \geq 0 \quad \textit{for} \quad 0 \leq \mu_n < \Omega;$$

*then the "partial sum"*

$$a_0 + 2 \sum_{\mu_n < \omega} a(\mu_n) \cos \mu_n x, \qquad \omega < \Omega \tag{22}$$

*of the Fourier expansion* (21) *converges absolutely in* $-\infty < x < \infty$; *its sum function* $\phi_\omega(x)$, *an almost periodic function whose Fourier expansion is identical with* (22), *satisfies the inequality*

$$|\phi_\omega(x)| \leq \frac{U}{1 - \omega/\Omega}. \tag{23}$$

For the proof, we associate with (21) the "Fejér-polynomials" $\Sigma_q(x)$, defined, for $q \geq 1$, by

$$\Sigma_q(x) = M \left\{ \phi(t) \sum_{\nu_1=-P}^{P} \sum_{\nu_2=-P}^{P} \cdots \sum_{\nu_q=-P}^{P} \left(1 - \frac{|\nu_1|}{P}\right) \right. \tag{24}$$
$$\left. \cdots \left(1 - \frac{|\nu_q|}{P}\right) \cos t \, \frac{\nu_1\rho_1 + \cdots + \nu_q\rho_q}{Q} \cos x \, \frac{\nu_1\rho_1 + \cdots + \nu_q\rho_q}{Q} \right\},$$

where $Q = q!$, $P = qQ = q \cdot q!$. If the sequence $(\rho_\nu)$ consists of $q_0$ terms only, we put $\rho_\nu = 0$ for $\nu > q_0$. By virtue of (20) and the linear independence of the $\rho_\nu$'s, we can write (24) in the form

$$\Sigma_q(x) = a_0 + \sum k_n^{(q)} a_n \cos \lambda_n x, \tag{25}$$

where

$$k_n^{(q)} = \prod_{a=1}^{q} \left(1 - \frac{|\nu_a|}{P}\right), \text{ whenever } 0 < \lambda_n = \frac{1}{Q} \sum_{a=1}^{q} \nu_a \rho_a, \tag{26}$$

the $\nu_a$ being integers with $|\nu_a| < P$, while $k_n^{(q)} = 0$ for other values. Obviously $0 \leq k_n^{(q)} < 1$. We shall prove

$$\lim_{q\to\infty} k_n^{(q)} = 1, \text{ whenever } \lambda_n \subset \{\mu_\nu\}. \tag{27}$$

In fact, we then have

$$\lambda_n = \sum_{\alpha=1}^{h} r_\alpha \rho_\alpha = \sum_{\alpha=1}^{h} \frac{u_\alpha}{v_\alpha} \rho_\alpha, \qquad u_\alpha \text{ and } v_\alpha > 0, \text{ integers.}$$

Hence, for

$$q > \max(v_1, \cdots, v_h, h, |r_1|, \cdots, |r_h|),$$

$$\lambda_n = \frac{\nu_1}{Q}\rho_1 + \cdots + \frac{\nu_h}{Q}\rho_h + \frac{0}{Q}\rho_{h+1} + \cdots + \frac{0}{Q}\rho_q,$$

where $\nu_\alpha = Q r_\alpha$ for $1 \leq \alpha \leq h$. Consequently, from (26),

$$k_n^{(q)} = \prod_{\alpha=1}^{h} \left(1 - \frac{Q\,|r_\alpha|}{p}\right) = \prod_{\alpha=1}^{h} \left(1 - \frac{|r_\alpha|}{q}\right) \uparrow 1, \text{ for } q \to \infty.$$

Finally we prove

$$\Sigma_q(x) \leq U. \tag{28}$$

This follows from

$$\begin{aligned}\Sigma_q(x) = M\Big\{ \phi(t) \sum_{\nu_1, \ldots, \nu_q}^{-P,P} \prod_{1}^{q} \left(1 - \frac{|\nu_\alpha|}{P}\right) \tfrac{1}{4}\Big[&\exp\Big(i(t+x)\sum_{1}^{q} \nu_\alpha\rho_\alpha/Q\Big) \\ &+ \exp\Big(-i(t+x)\sum_{1}^{q} \nu_\alpha\rho_\alpha/Q\Big) + \exp\Big(i(t-x)\sum_{1}^{q} \nu_\alpha\rho_\alpha/Q\Big) \\ &+ \exp\Big(-i(t-x)\sum_{1}^{q} \nu_\alpha\rho_\alpha/Q\Big)\Big]\Big\} \\ = M\{\phi(t)\tfrac{1}{2}[K_q(t+x) + K_q(t-x)]\},&\end{aligned}$$

where

$$\begin{aligned}K_q(t) &= \sum_{\nu_1, \ldots, \nu_q}^{-P,P} \prod_{1}^{q} \left(1 - \frac{|\nu_\alpha|}{P}\right) \exp\Big(it\sum_{1}^{q} \nu_\alpha\rho_\alpha/Q\Big) \\ &= \prod_{\alpha=1}^{q} \sum_{\nu_\alpha=-P}^{P} \left(1 - \frac{|\nu_\alpha|}{P}\right) \exp(i\nu_\alpha\rho_\alpha t/Q) \\ &= \prod_{\alpha=1}^{q} \frac{1}{P} \left[\frac{\sin\dfrac{P}{2}\dfrac{\rho_\alpha}{Q}t}{\sin\dfrac{\rho_\alpha}{2Q}}\right]^2.\end{aligned}$$

From this we see that

$$\Sigma_q(x) \leq UM\{\tfrac{1}{2}[K_q(t+x) + K_q(t-x)]\} = U,$$

and, in particular,

$$\Sigma_q(0) = a_0 + \sum k_n^{(q)} a_n \leq U.$$

We next define for any positive $\mu$ the "Riesz-mean"

$$R_\mu(x, q) = a_0 + \sum_{0 < \lambda_n < \mu} \left(1 - \frac{\lambda_n}{\mu}\right) k_n^{(q)} a_n \cos \lambda_n x$$

associated with the polynomial (25). Applying the formulas

$$\frac{2}{\pi}\int_0^\infty \cos 2kt \,\frac{\sin^2 \mu t}{\mu t^2}\, dt = \begin{cases} 1 - k/\mu & \text{if } \; 0 \leq k < \mu \\ 0 & \text{if } \; 0 < \mu \leq k \end{cases} \tag{29}$$

and using (25), we obtain

$$R_\mu(x, q) = \frac{2}{\pi}\int_0^\infty \tfrac{1}{2}[\Sigma_q(x + 2t) + \Sigma_q(x - 2t)]\,\frac{\sin^2 \mu t}{\mu t^2}\, dt.$$

From this and (28) we find

$$R_\mu(x, q) \leq \frac{2}{\pi}\, U \int_0^\infty \frac{\sin^2 \mu t}{\mu t^2}\, dt = U;$$

and, in particular,

$$R_\mu(0, q) = a_0 + \sum_{0 < \lambda_n < \mu}\left(1 - \frac{\lambda_n}{\mu}\right) k_n^{(q)} a_n \leq U.$$

Since every term is positive if $\mu < \Omega$, we get for an arbitrary $N$

$$a_0 + \sum_{0 < \lambda_n < \mu}^{n < N}\left(1 - \frac{\lambda_n}{\mu}\right) k_n^{(q)} a_n \leq U.$$

Now passing to the limit $q \to \infty$,

$$a_0 + 2\sum_{0 < \mu_m < \mu}^{m \leq M}\left(1 - \frac{\mu_m}{\mu}\right) a(\mu_m) \leq U, \qquad M = M(N),$$

and, a fortiori for $0 < \omega < \mu < \Omega$,

$$a_0 + 2\sum_{0 < \mu_m < \omega}^{m \leq M}\left(1 - \frac{\mu_m}{\mu}\right) a(\mu_m) \leq U.$$

Here $1 - \mu_m/\mu > 1 - \omega/\mu$; hence,

$$a_0 + 2\left(1 - \frac{\omega}{\mu}\right)\sum_{0 < \mu_m < \omega}^{m \leq M} a(\mu_m) \leq U,$$

and for $\mu \to \Omega$

$$a_0 + 2\sum_{0 < \mu_m < \omega}^{m \leq M} a(\mu_m) \leq \frac{U}{1 - \omega/\Omega}.$$

As $N \to \infty$, we have $M \to \infty$, and

$$a_0 + 2\sum_{0 < \mu_m < \omega} a(\mu_m) \leq \frac{U}{1 - \omega/\Omega}. \tag{30}$$

This gives us the inequality (23). Essentially this is Fekete's proof.

**6.** We now consider odd functions $\psi(-x) = -\psi(x)$, and the corresponding generalized sine-series. We assume again that

$$\sup_x \frac{1}{x}\int_0^x \psi^2(t)\,dt < \infty$$

and that

$$\lim_{x\to\infty}\frac{1}{x}\int_0^x \psi(t)\sin\lambda t\,dt = b(\lambda) = M\{\omega(t)\sin\lambda t\}$$

exists for all $\lambda > 0$. Then, again, $b(\lambda)$ vanishes, except at an enumerable set of $\lambda$-values. We denote in a certain order by $\lambda_1, \lambda_2, \cdots$ those $\lambda$ for which $b(\lambda) \neq 0$, and call the series

$$(32) \qquad \sum_1^\infty b_n \sin\lambda_n x \sim \psi(x) \quad \text{where} \quad b_n = 2b(\lambda_n),$$

the Fourier expansion of $\psi(x)$. We denote by $\rho_1, \rho_2, \cdots$ a subsequence of $\{\lambda_n\}$, consisting of linearly independent numbers, and by $\{\mu_n\}$ the subsequence of those $\lambda_n$, which can be represented in the form

$$\lambda_n = \sum_{\nu=1}^h r_\nu\rho_\nu, \quad h = h(n), \quad \text{where} \quad r_\nu = r_\nu(n) \quad \text{are rational.}$$

Again $0 < \omega < \Omega$, and we suppose

$$b(\mu_n) \geq 0 \quad \text{for} \quad 0 < \mu_n < \Omega.$$

If we associate with (32) the polynomials

$$T_q(x) = M\left\{\psi(t)\sum_{\nu_1=-P}^{P}\cdots\sum_{\nu_q=-P}^{P}\left(1-\frac{|\nu_1|}{P}\right)\right.$$
$$\left.\cdots\left(1-\frac{|\nu_q|}{P}\right)\sin t\,\frac{\nu_1\rho_1+\cdots+\nu_q\rho_q}{Q}\sin x\,\frac{\nu_1\rho_1+\cdots+\nu_q\rho_q}{Q}\right\}, \quad q \geq 1,$$

then by virtue of (31) we can write

$$T_q(x) = \sum k_n{}^{(q)} b_n \sin\lambda_n x,$$

where

$$k_n{}^{(q)} = \prod_{a=1}^q\left(1-\frac{|\nu_a|}{P}\right) \text{ whenever } 0 < \lambda_n = \frac{1}{Q}\sum_{a=1}^q \nu_a\rho_a$$

and $k_n{}^{(q)} = 0$ for all other cases. We now have

$$T_q(x) = M\{\psi(t)\tfrac{1}{2}[K_q(t-x) + K_q(t+x)]\},$$

and on assuming

$$|\psi(t)| \leq U, \qquad t > 0$$

we get

$$(33) \qquad |T_q(x)| \leq U, \qquad x > 0.$$

Again, introducing the polynomials

$$S_\mu(x,q)=\sum_{0<\lambda_n\leq\omega}\left(1-\frac{\lambda_n}{\mu}\right)k_n^{(q)}b_n\sin\lambda_n x,\qquad \mu<\Omega,$$

and applying formulas (29) we obtain

$$S_\mu(x,q)=\frac{2}{\pi}\int_0^\infty \tfrac{1}{2}[T_q(x+2t)-T_q(x-2t)]\,\frac{\sin^2\mu t}{\mu t^2}\,dt.$$

From this and (33) follows

$$|S_\mu(x,q)|\leq U,\qquad x>0. \tag{34}$$

At this point we simplify Fekete's argument, replacing a theorem of S. Bernstein by the following

Lemma. *Given a generalized sine-polynomial*

$$S(x)=\sum_1^n b_\nu\sin\lambda_\nu x,\qquad \lambda_\nu>0,\qquad (\nu=1,2,\cdots,n),$$

*assume*

$$b_\nu>0,\ (\nu=1,2,\cdots,n),\ \textit{and}\ S(x)\leq U\ \textit{for}\ 0<x<\xi.$$

*Then*

$$\sum_{\lambda_\nu\leq\omega}\lambda_\nu b_\nu\leq\frac{\alpha}{\sin^2\alpha}\,\omega U<1.38\omega U,\ \text{for}\ \omega\geq\frac{2\alpha}{\xi},$$

*where $\alpha$ is defined in Theorem 2.*

For the proof consider

$$\int_0^x S(t)\,dt=\sum_1^n b_\nu\frac{1-\cos\lambda_\nu x}{\lambda_\nu}=\tfrac{1}{2}x^2\sum_1^n\lambda_\nu b_\nu\left(\frac{\sin\frac{1}{2}\lambda_\nu x}{\frac{1}{2}\lambda_\nu x}\right)^2,$$

from which we find

$$\tfrac{1}{2}x^2\sum_{\lambda_\nu\leq\omega}\lambda_\nu b_\nu\left(\frac{\sin\frac{1}{2}\lambda_\nu x}{\frac{1}{2}\lambda_\nu x}\right)^2\leq\int_0^x S(t)\,dt\leq xU.$$

Since $\omega x\leq\pi$,

$$\frac{\sin\frac{1}{2}\lambda_\nu x}{\frac{1}{2}\lambda_\nu x}\geq\frac{\sin\frac{1}{2}\omega x}{\frac{1}{2}\omega x},\qquad \lambda_\nu\leq\omega,$$

and

$$\left(\frac{\sin\frac{1}{2}\omega x}{\frac{1}{2}\omega x}\right)^2\frac{x}{2}\sum_{\lambda_\nu\leq\omega}\lambda_\nu b_\nu\leq U.$$

Finally, choosing $x=2\alpha/\omega<\pi/\omega$, we have

$$\sum_{\lambda_\nu\leq\omega}\lambda_\nu b_\nu\leq\frac{\alpha}{\sin^2\alpha}\,\omega U,\ \text{for}\ \frac{2\alpha}{\omega}\leq\xi.$$

Applying this lemma to the polynomial $S_\mu(x,q)$, we get, from (34),

$$(35)\qquad \sum_{0<\lambda_n\leq\omega}\left(1-\frac{\lambda_n}{\mu}\right)k_n^{(q)}\lambda_n b_n \leq \frac{\alpha}{\sin^2\alpha}\,\omega U, \qquad 0<\omega<\Omega,$$

and passing to the limit $q \to \infty$,

$$2\sum_{0<\mu_m\leq\omega}\left(1-\frac{\mu_m}{\mu}\right)\mu_m b(\mu_m) \leq \frac{\alpha}{\sin^2\alpha}\,\omega U.$$

But $1-\mu_m/\mu \geq 1-\omega/\mu$, hence

$$2\sum_{0<\mu_m\leq\omega}\mu_m b(\mu_m) \leq \frac{\omega U\dfrac{\alpha}{\sin^2\alpha}}{1-\dfrac{\omega}{\mu}};$$

and, as $\mu \to \Omega$, this takes the form

$$(36)\qquad 2\sum_{0<\mu_m\leq\omega}\mu_m b(\mu_m) \leq \frac{\omega U\dfrac{\alpha}{\sin^2\alpha}}{1-\dfrac{\omega}{\Omega}},$$

which is sharper than Fekete's inequality

$$2\sum_{0<\mu_m\leq\omega}\mu_m b(\mu_m) \leq \frac{\Omega U}{1-\dfrac{\omega}{\Omega}},$$

if we restrict ourselves to $\omega\,\dfrac{\alpha}{\sin^2\alpha}<\Omega$.

From (36) it follows that

$$\Big|\sum_{0<\lambda_n\leq\omega}k_n^{(q)}\frac{\lambda_n}{\omega}\,b_n\sin\lambda_n x\Big| \leq \frac{U\dfrac{\alpha}{\sin^2\alpha}}{1-\dfrac{\omega}{\Omega}},$$

and from (34), for $\mu=\omega$,

$$\Big|\sum_{0<\lambda_n\leq\omega}k_n^{(q)}\left(1-\frac{\lambda_n}{\omega}\right)b_n\sin\lambda_n x\Big| \leq U.$$

Thus, writing $p_m^{(q)}$ for $k_n^{(q)}$ if $\lambda_n=\mu_m$,

$$2\Big|\sum_{0<\mu_m\leq\omega}p_m^{(q)}b(\mu_m)\sin\mu_m x\Big| \leq U\left(1+\frac{\alpha}{\sin^2\alpha\left(1-\dfrac{\omega}{\Omega}\right)}\right).$$

From (36) follows, as Fekete proved, that $\sum_{0<\mu_m\leq\omega}b(\mu_m)\sin\mu_m x$ converges absolutely, hence by (27)

$$(37)\qquad \lim_{q\to\infty}2\sum_{0<\mu_m\leq\omega}p_m^{(q)}b(\mu_m)\sin\mu_m x = 2\sum_{0<\mu_m\leq\omega}b(\mu_m)\sin\mu_m x = \psi_\omega(x),$$

$$(38)\qquad |\psi_\omega(x)| \leq U\left(1+\frac{\alpha}{\sin^2\alpha\left(1-\dfrac{\omega}{\Omega}\right)}\right).$$

If $\{\rho_k\}$ is a basic sequence of $\{\lambda_k\}$, it follows that

$$\psi_\omega(x) = \sum_{0<\lambda_n\leq\omega} b_n \sin \lambda_n x, \qquad 0 < \omega < \Omega,$$

and this series converges absolutely. If, in particular, all $a_n \geq 0$ and all $b_n \geq 0$, then $\Omega$ is arbitrary, and, letting $\Omega \to \infty$, from (22) and (30) we find

$$\sum_{0<\lambda_n\leq\omega} a_n \leq U, \qquad |\psi_\omega(x)| \leq U\left[1 + \frac{\alpha}{\sin^2 \alpha}\right].$$

Hence for any real function $f(x)$ which satisfies

$$|f(x)| \leq U, \qquad -\infty < x < +\infty$$
$$M\{f(t)e^{-i\lambda t}\} = c(\lambda); \qquad c(\lambda_n) = \tfrac{1}{2}(a_n - ib_n),$$

we have

$$|a_0 + \sum_{0<\lambda_n\leq\omega} (a_n \cos nx + b_n \sin nx)| \leq U\left[2 + \frac{\alpha}{\sin^2 \alpha}\right].$$

The fact that the functions $\phi_\omega(x)$, $\psi_\omega(x)$ are u. a. p. and that their Fourier expansions are (17) and (29) respectively follows in exactly the same way as in Fekete's theorem.

Analogous argument can be applied to derive corresponding theorems for trigonometric integrals (cf. Szász [5]) and also to improve upon some results concerning Fourier series with $na_n \geq -K$, $nb_n \geq -K$ and more general classes (cf. Szász [4]).

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